COMBINATORICS OF THE $\widehat{sl}_2$ SPACES OF COINVARIANTS III

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Abstract. We give the fermionic character formulas for the spaces of coinvariants obtained from level k integrable representations of $\widehat{\mathfrak{sl}}_2$. We establish the functional realization of the spaces dual to the coinvariant spaces. We parameterize functions in the dual spaces by rigged partitions, and prove the recursion relations for the sets of rigged partitions.

1. Introduction

1.1. **Coinvariant spaces of $\widehat{\mathfrak{sl}}_2$.** Let $\mathfrak{a}$ be a Lie subalgebra of a Lie algebra $\mathfrak{g}$, and L a $\mathfrak{g}$-module. The quotient space $L/\mathfrak{a}L$ is called the space of coinvariants of L with respect to $\mathfrak{a}$. In [FKLMM1, FKLMM2] we studied spaces of coinvariants for integrable $\widehat{\mathfrak{sl}}_2$-modules. The present paper is Part III of the series. We make extensive use of the results of the pervious papers.

In this paper, we consider the following special case of the coinvariant. Let e_i, f_i, h_i ($i \in \mathbb{Z}$) be the loop generators of $\widehat{\mathfrak{sl}}_2$, and $\mathfrak{a} = \mathfrak{a}^{(M,N)}$ the subalgebra generated by $\{e_i(i \geq M); f_i(i \geq N)\}$. Let $L = L_{k,l}$ be the level-k integrable highest weight $\widehat{\mathfrak{sl}}_2$-module with highest weight $(k-l)\Lambda_0 + l\Lambda_1$. We are interested in the coinvariant

$$L_{k,l}^{(M,N)} = L_{k,l}/\mathfrak{a}^{(M,N)}L_{k,l}. \tag{1.1}$$

The main result of [FKLMM2] was a theorem about the dimension of this space, which we showed is given by the Verlinde rule:

Theorem 1.1.1. *For $M, N \geq 0$,*

$$\dim L_{k,l}^{(M,N)} = \#\left(\mathcal{P}_{k,l}^{M+N}\right), \tag{1.2}$$

where $\mathcal{P}_{k,l}^{N}$ is the set of level-k admissible paths of length N and weight l. (see [FKLMM2] *for the precise definition).*

In fact, the coinvariant space inherits a graded structure from the integrable module $L_{k,l}$. Let d denote the homogeneous degree element of $\widehat{\mathfrak{sl}}_2$, $[d, x_i] = ix_i$ for $x \in \mathfrak{sl}_2$ and define the Hilbert polynomial or character of the coinvariant space to be

$$\chi_{k,l}^{(M,N)}(z,q) = \mathrm{trace}_{L_{k,l}^{(M,N)}} q^d z^{h_0}$$

where $h_0 = h \in \mathfrak{sl}_2$. In [FKLMM2] we used a recursion relation for such characters to prove Theorem 1.1.1. The purpose of this paper is to derive explicit formulas for these polynomials. It turns out that our procedure naturally results in fermionic formulas for the characters. See [FS, St] for some related formulas in the special case $l = 0$.

1.2. **The Heisenberg loop algebra and coinvariants.** In order to study the dimension of the coinvariant, in [FKLMM2] we introduced the simpler coinvariants associated with modules of the Heisenberg loop algebra.

Let $\mathfrak{H}$ be the three dimensional Heisenberg algebra with generators e, f, h and relations $[e,f]=h$ and h central (note that we use the same notation for the generators of $\mathfrak{sl}_2$, but the relations are different; it should be clear from the context which algebra the generators belong to). Let $\widetilde{\mathfrak{H}}$ be the algebra of loops into $\mathfrak{H}$, generated by $\{e_i, f_i, h_i; i\in\mathbb{Z}\}$ with relations

$$[e_i,f_j]=h_{i+j},\qquad [h_i,e_j]=[h_i,f_j]=0.$$

Note, that in contrast to $\widehat{\mathfrak{s}l}_2$, $\widetilde{\mathfrak{H}}$ has a triple-grading, with degrees defined by

$$\deg e_i=(1,0,i),\quad \deg f_i=(0,1,i),\quad \deg h_i=(1,1,i). \tag{1.3}$$

Let $W_k[l_1,l_2,l_3]$ be the k-restricted $\widetilde{\mathfrak{H}}$-module (see (3.7) for the definition). It is the analog of the level-k $\widehat{\mathfrak{sl}}_2$-modules, although it is not irreducible. It turns out that there is a simple relationship between the characters of these modules and those of $L_{k,l}$.

We consider the coinvariants of $W_k[l_1,l_2,l_3]$ with respect to the $\widetilde{\mathfrak{H}}$ subalgebras $\mathfrak{a}=\mathfrak{a}^{(M,N)}$ generated by the set of elements $\{e_i(i\geq M); f_i(i\geq N)\}$:

$$W_k^{(M,N)}[l_1,l_2,l_3]=W_k[l_1,l_2,l_3]/\mathfrak{a}^{(M,N)}W_k[l_1,l_2,l_3].$$

(In this section, we assume $M,N\geq 1$, but in the main text, we treat $M,N\geq 0$.) The $\widetilde{\mathfrak{H}}$-modules and coinvariants inherit the triple-grading (1.3), and we define the character by

$$\chi_k^{(M,N)}[l_1,l_2,l_3](z_1,z_2,q)=\sum_{m,n,d}\dim(W_k^{(M,N)}[l_1,l_2,l_3]_{m,n,d})\; z_1^m z_2^n q^d,$$

where $W_k^{(M,N)}[l_1,l_2,l_3]_{m,n,d}$ is the subspace of degree (m,n,d).

In [FKLMM2], we showed that $\widehat{\mathfrak{sl}}_2$-coinvariants and $\widetilde{\mathfrak{H}}$-coinvariants are closely related, and that $\chi_{k,l}^{(M,N)}$ is given in terms of $\chi_k^{(M,N)}[l_1,l_2]=\chi_k^{(M,N)}[l_1,l_2,\min(l_1,l_2)]$:

$$z^l\chi_{k,l}^{(M,N)}(q,z)=\chi_k^{(M+1,N)}[l,k-l](q^{-2}z^2,z^{-2},q)-q\chi_k^{(M+1,N)}[l-1,k-l-1](q^{-2}z^2,z^{-2},q).$$

The key property of $\chi_k^{(M,N)}[l_1,l_2,l_3]$ used in the proof of Theorem 1.1.1, is that it satisfies the following recursion relation with respect to (M,N) (see Theorem 6.1.5 of [FKLMM2]:

Theorem 1.2.1.

$$\chi_k^{(M,N)}[l_1,l_2,l_3](z_1,z_2,q)=\sum_{\substack{0\leq a\leq l_3\\ 0\leq c\leq l_2-a}} z_1^a z_2^{a+c}q^{a+c}\;\chi_k^{(M,N-1)}[l_1',l_2',l_3'](z_1,qz_2,q) \tag{1.4}$$

where $l_1'=\min(l_1+c-a,k-a),\quad l_2'=k-c,\qquad l_3'=l_1'+l_2'-k.$

In this paper, we give an explicit formula for the characters $\chi_k^{(M,N)}[l_1,l_2,l_3]$ (see Theorem 3.6.2). These formulas have a fermionic form in the sense of [KKMM].

1.3. **Functional realization of dual spaces.** The basic idea in deriving closed forms for the characters is to consider the function spaces $W_k^{(M,N)}[l_1,l_2,l_3]^*$ dual to $W_k^{(M,N)}[l_1,l_2,l_3]$. The defining relations for $\widetilde{\mathfrak{H}}$ are simpler than those for $\widehat{\mathfrak{sl}}_2$ because they respect the grading (1.3). As a consequence, for each fixed m,n, the space $W_k^{(M,N)}[l_1,l_2,l_3]^*_{m,n}$ can be realized as a subspace of the space of rational functions $F(x_1,\dots,x_m;y_1,\dots,y_n)$, symmetric in each set $\{x_1,\dots,x_m\}$ and $\{y_1,\dots,y_n\}$ separately, having at most simple poles when $x_i=y_j$ and zeros on the submanifolds $x_i=x_j=y_l$ $(i\neq j)$ and $x_i=y_j=y_l$ $(j\neq l)$. The dual space is characterized by the vanishing of functions F on certain submanifolds depending on k,l_1,l_2,l_3. For example, the restriction related to the level k reads as

$$F=0 \quad \text{if } x_1=\cdots=x_{k+1} \text{ or } y_1=\cdots=y_{k+1}. \tag{1.5}$$

(See section 3.2 for the full definition.) Because of the high codimensionality of these submanifolds, it is not possible to immediately deduce the formulas for the characters, and it is necessary to introduce a filtration on the dual space, such that adjoint graded spaces are isomorphic simply to spaces of symmetric functions, and thus have simple characters. We follow [FS] in this process.

Let

$$\mu=\left(k^{m_k},(k-1)^{m_{k-1}},\dots,1^{m_1}\right),\quad \nu=\left(k^{n_k},(k-1)^{n_{k-1}},\dots,1^{n_1}\right) \tag{1.6}$$

be level-k restricted partitions of m and n, respectively, so that $\sum_\alpha \alpha m_\alpha=m$ and $\sum_\alpha \alpha n_\alpha=n$. We consider the following family of submanifolds

$$\mathcal{M}_{\mu,\nu}: x^{(\alpha)}_{i,1}=\cdots=x^{(\alpha)}_{i,\alpha}\quad (1\le\alpha\le k; 1\le i\le m_\alpha),\quad y^{(\alpha)}_{i,1}=\cdots=y^{(\alpha)}_{i,\alpha}\quad (1\le\alpha\le k; 1\le i\le n_\alpha), \tag{1.7}$$

where the sets of variables $\{x_j\}$, $\{y_j\}$ are relabeled $\{x^{(\alpha)}_{i,l}\}$ and $\{y^{(\alpha)}_{i,l}\}$, respectively.

A subspace $\mathcal{F}_{\mu,\nu}\subset W_k[l_1,l_2,l_3]^*_{m,n}$ is the subspace of functions vanishing on the submanifolds $\mathcal{M}_{\mu,\nu}$. Using lexicographic ordering on partitions, these give a filtration of the dual space, and the adjoint graded space to this filtration has a simple structure. For example, if $l_3=\min(l_1,l_2)$, the graded component corresponding to (μ,ν) is spanned by the set of all symmetric polynomials on $\mathcal{M}_{\mu,\nu}$. More precisely, we identify the (μ,ν)-graded component with the space of functions of the form $G_{\mu,\nu}g$, where $G_{\mu,\nu}$ is a fixed rational function depending only on μ,ν and g is an arbitrary polynomial in the variables $\{x^{(\alpha)}_i\}_{1\le i\le m_\alpha}$ and $\{y^{(\alpha)}_i\}_{1\le i\le n_\alpha}$, $1\le\alpha\le k$, symmetric under the exchange of variables with the same superscript α, $x^{(\alpha)}_i\leftrightarrow x^{(\alpha)}_j$ or $y^{(\alpha)}_i\leftrightarrow y^{(\alpha)}_j$. This space has a basis $\mathrm{Sym}(\prod_{\alpha,i}(x^{(\alpha)}_i)^{r^{(\alpha)}_i}(y^{(\alpha)}_i)^{s^{(\alpha)}_i})$, where for each α, $r^{(\alpha)}=\{r^{(\alpha)}_{1\le i\le m_\alpha}\}$ and $s^{(\alpha)}=\{s^{(\alpha)}_{1\le i\le n_\alpha}\}$ are sets of integers satisfying $r^{(\alpha)}_1\ge\cdots\ge r^{(\alpha)}_{m_\alpha}\ge 0$ and $s^{(\alpha)}_1\ge\cdots\ge s^{(\alpha)}_{n_\alpha}\ge 0$, respectively.

These basis elements are in one to one correspondence with combinatorial data $(\mu,r;\nu,s)$ called *rigged partitions*, introduced in [KKR, KR]. The set of non-negative integers r is called a rigging of the partition μ.

If $l_3<\min(l_1,l_2)$, there is an additional restrictions for the riggings from below,

$$r^{(\alpha)}_i+s^{(\beta)}_j\ge\min(\alpha,\beta)-\max(\alpha-l_1,0)-\max(\beta-l_2,0)-l_3. \tag{1.8}$$

The space dual to the coinvariant, $W_k^{(M,N)}[l_1,l_2,l_3]^*$, is the subspace of functions F which satisfy the degree restrictions

$$\deg_{x_1} F < M, \quad \deg_{y_1} F < N.$$

We will show that the degree restrictions translates to conditions for the riggings r and s of the form

$$r_i^{(\alpha)} \le P_{\mu,\nu}^{(M)}[l_1]_\alpha, \quad s_i^{(\alpha)} \le Q_{\mu,\nu}^{(N)}[l_2]_\alpha, \tag{1.9}$$

where the vacancy numbers $P_{\mu,\nu}^{(M)}[l_1]$, $Q_{\mu,\nu}^{(N)}[l_2]$ are defined in equations (2.6), (2.7).

Our final result is that the adjoint graded space of $W_k^{(M,N)}[l_1,l_2,l_3]^*_{m,n}$ has a basis labeled by pairs of rigged partitions $(\mu,r;\nu,s)$ with the restrictions on the riggings of the form (1.8) and (1.9). Denote the set of such rigged partitions by $R_{m,n}^{(M,N)}[l_1,l_2,l_3]$. Because of Theorem 1.2.1, one can expect that there is an inductive construction of $R_{m,n}^{(M,N)}[l_1,l_2,l_3]$ from $R_{m-a,n-a-c}^{(M,N-1)}[l_1',l_2',l_1'+l_2'-k]$. In fact, this is true, and we will describe it explicitly.

The logical ordering of this paper is somewhat different. We prove directly that the evaluation map which maps the space of functions of the form F to the space of functions spanned by $G_{\mu,\nu}g$ is injective. However, we do not have a simple direct proof that it is surjective. Instead, we construct $R_{m,n}^{(M,N)}[l_1,l_2,l_3]$ inductively, and this assures the surjectivity by dimension counting arguments.

The plan of paper is as follows. In Section 2 we give preliminaries on rigged partitions and state the main recursion theorem (Theorem 2.2.1). In Section 3, we construct the functional realization of dual spaces, their filtrations and describe the adjoint graded spaces. We also give the resulting fermionic formulas for the characters. Sections 4, 5 and 6 are devoted to the proof of Theorem 2.2.1. The arguments in these sections are purely combinatorial. In Section 4, we define admissible pairs (I,J) of index sets belonging to $\{1,\dots,k\}$. Then, we define two types of subsets of rigged partitions indexed by admissible pairs, the lower and upper subsets. We construct a bijection from the upper to the lower subsets indexed by the same pair (I,J). In Sections 5 and 6, we give the decompositions of the set of rigged partitions for (M,N) by the lower subsets, and that for $(M,N-1)$ by the upper subsets, respectively. This completes the proof of Theorem 2.2.1.

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2. Rigged partitions and the main recursion theorem

We define level restricted rigged partitions, and state the main recursion theorem for sets of rigged partitions, Theorem 2.2.1, together with an outline the proof.

2.1. **Rigged partitions and vacancy numbers.**

Let $k \in \mathbb{Z}_{\ge 1}$, $m \in \mathbb{Z}_{\ge 0}$ and $I_k = \{1,2,...,k\}$. Let μ be a level-k restricted partition of m, that is

$$\mu = (k^{m_k},\dots,2^{m_2},1^{m_1}), \qquad \sum_{\alpha=1}^k \alpha m_\alpha = m, \tag{2.1}$$

We denote by $m_\alpha(\mu)$ the number of rows of length α in the partition (or Young diagram) μ.

A rigging of μ is a set of integers $r = \{r_i^{(\alpha)}\}_{\alpha\in I_k,\ 1\le i\le m_\alpha(\mu)}$ such that

$$r_1^{(\alpha)} \ge \dots \ge r_{m_\alpha(\mu)}^{(\alpha)} \ge 0 \quad (\alpha \in I_k). \tag{2.2}$$

A partition with a rigging, (μ, r), is called a rigged partition. Denote by R_m the set of all such level-k restricted rigged partitions of m. We set $R_{m,n} = R_m \times R_n$.

Let l_1, l_2, l_3 be integers satisfying

$$0 \le l_1, l_2 \le k, \quad 0 \le l_3 \le \min(l_1, l_2). \tag{2.3}$$

Define

$$\begin{aligned}\tau^{(\alpha,\beta)}[l_1, l_2, l_3] &= \min(\alpha, \beta, l_1, l_2, l_1 + \beta - \alpha, l_2 + \alpha - \beta, l_1 + l_2 - \alpha, l_1 + l_2 - \beta) - l_3 \\ &= \min(\alpha, \beta) - (\alpha - l_1)^+ - (\beta - l_2)^+ - l_3, \end{aligned} \tag{2.4}$$

where $x^+ = \max(x, 0), \quad x^- = \max(-x, 0)$. We define a subset of $R_{m,n}$ where the lower bounds of the riggings are restricted by (2.4):

$$R_{m,n}[l_1, l_2, l_3] = \Big\{(\mu, r; \nu, s) \in R_{m,n} \ ; \ r^{(\alpha)}_{m_\alpha(\mu)} + s^{(\beta)}_{m_\beta(\mu)} \ge \tau^{(\alpha,\beta)}[l_1, l_2, l_3]\Big\} \tag{2.5}$$

Since $\tau^{(\alpha,\beta)}[l_1, l_2, \min(l_1, l_2)] \le 0$, there is no restriction in this case, and $R_{m,n}[l_1, l_2, \min(l_1, l_2)] = R_{m,n}$

Let M, N be non-negative integers. Define vectors of vacancy numbers $P^{(M)}_{\mu,\nu}[l_1], Q^{(N)}_{\mu,\nu}[l_2] \in \mathbb{Z}^k$, where

$$P^{(M)}_{\mu,\nu}[l]_\alpha = \alpha M - (\alpha - l)^+ + \sum_{\beta=1}^{k} \min(\alpha, \beta)(m_\beta(\nu) - 2m_\beta(\mu)), \tag{2.6}$$

$$Q^{(N)}_{\mu,\nu}[l]_\alpha = P^{(N)}_{\nu,\mu}[l]_\alpha. \tag{2.7}$$

We define the subset $R^{(M,N)}_{m,n}[l_1, l_2] \subset R_{m,n}$:

$$R^{(M,N)}_{m,n}[l_1, l_2] = \{(\mu, r; \nu, s) \in R_{m,n} \quad : \quad P^{(M)}_{\mu,\nu}[l_1]_\alpha, Q^{(N)}_{\mu,\nu}[l_2]_\alpha \ge 0, \tag{2.8}$$

$$r^{(\alpha)}_1 \le P^{(M)}_{\mu,\nu}[l_1]_\alpha, \ s^{(\alpha)}_1 \le Q^{(N)}_{\mu,\nu}[l_2]_\alpha\}. \tag{2.9}$$

The first condition, (2.8), is non-trivial only in the case $m_\alpha(\mu) = 0$ or $m_\alpha(\nu) = 0$. Otherwise, it follows from (2.9). However, see Proposition 2.1.1 for the actual implication of this conditions.

Finally define the set

$$R^{(M,N)}_{m,n}[l_1, l_2, l_3] = R^{(M,N)}_{m,n}[l_1, l_2] \cap R_{m,n}[l_1, l_2, l_3]. \tag{2.10}$$

It is defined for negative values of m, n by

$$R^{(M,N)}_{m,n}[l_1, l_2, l_3] = \emptyset, \qquad m < 0 \ \text{ or } \ n < 0.$$

Before passing, we prove

Proposition 2.1.1. *Suppose that the conditions* (2.9) *hold. Then, the conditions* (2.8) *are equivalent to the following requirements:*

$$\textit{If } M = 0 \textit{ then } n - 2m \ge k - l_1; \tag{2.11}$$

$$\textit{If } N = 0 \textit{ then } m - 2n \ge k - l_2. \tag{2.12}$$

Namely, it is enough to require the conditions (2.8) *only for the cases* $M = 0, \alpha = k$ *and* $N = 0, \alpha = k$.

Proof. In the following, when we write a condition concerning the 0-th component of a k vector (e.g., the case $i=0$ for $P^{(M)}_{\mu,\nu}[l_1]_i\geq 0$ in the next paragraph or $P_i\geq\rho_i$ in the proof of Lemma 4.2.2), we mean that the condition is void.

First we prove that if $M\geq 1$ the condition $P^{(M)}_{\mu,\nu}[l_1]_k\geq 0$ follows from (2.9). Suppose otherwise, there exists $0\leq i\leq k-1$ such that $P^{(M)}_{\mu,\nu}[l_1]_i\geq 0$ and $m_{i+1}=\cdots=m_k=0$. (Here, $m_\alpha=m_\alpha(\mu), n_\alpha=m_\alpha(\nu)$.) Then, we have

$$\begin{aligned}
0 &> P^{(M)}_{\mu,\nu}[l_1]_k\\
&= P^{(M)}_{\mu,\nu}[l_1]_i+(k-i)M+(i-l_1)^+-(k-l_1)+\sum_{\beta\geq i+1}(\beta-i)n_\beta\\
&\geq 0,
\end{aligned}$$

which is a contradiction. Similarly, if $N\geq 1$ the condition $Q^{(N)}_{\mu,\nu}[l_2]_k\geq 0$ follows from (2.9).

Now we will prove that for all $M\geq 0$, the conditions $P^{(M)}_{\mu,\nu}[l_1]_\alpha\geq 0$ for $1\leq\alpha\leq k-1$ follow from $P^{(M)}_{\mu,\nu}[l_1]_k\geq 0$. (The proof is similar for $Q^{(N)}_{\mu,\nu}[l_2]_\alpha$.)

Suppose otherwise, there exists i and j such that $0\leq i<j-1\leq k-1$, $P^{(M)}_{\mu,\nu}[l_1]_i\geq 0$, $P^{(M)}_{\mu,\nu}[l_1]_j\geq 0$ and $P^{(M)}_{\mu,\nu}[l_1]_\alpha<0$ (and thereby $m_\alpha=0$) for $i+1\leq\alpha\leq j-1$. Set $p=\frac{1}{j-i}$ so that we have $pi+(1-p)j=j-1$. Then we have

$$\begin{aligned}
0 &\leq pP^{(M)}_{\mu,\nu}[l_1]_i+(1-p)P^{(M)}_{\mu,\nu}[l_1]_j\\
&= P^{(M)}_{\mu,\nu}[l_1]_{j-1}+(j-1-l_1)^+-p(i-l_1)^+-(1-p)(j-l_1)^+\\
&\qquad+\sum_\beta\Big(p\min(i,\beta)+(1-p)\min(j,\beta)-\min(j-1,\beta)\Big)n_\beta\\
&= P^{(M)}_{\mu,\nu}[l_1]_{j-1}-\theta(i+1\leq l_1\leq j-1)(l_1-i)p-\sum_{i+1\leq\beta\leq j-1}(\beta-i)pn_\beta\\
&< 0,
\end{aligned}\tag{2.13}$$

which is a contradiction. Here we used the notation

$$\theta(*)=\begin{cases}1 & \text{if } * \text{ is true;}\\ 0 & \text{if } * \text{ is false.}\end{cases}\tag{2.14}$$

□

Proposition 2.1.1 implies

Corollary 2.1.2. *For $(M,N)=(0,0)$ we have*

$$R^{(0,0)}_{m,n}[l_1,l_2,l_3]=\begin{cases}\{(\emptyset,\emptyset;\emptyset,\emptyset)\} & \text{if } l_1=l_2=k \text{ and } m=n=0;\\ \emptyset & \text{otherwise.}\end{cases}\tag{2.15}$$

2.2. **Recursion Theorem for rigged partitions.** We state the main theorem on recursion.

Theorem 2.2.1. *The cardinalities of the sets of the rigged partitions satisfy the following relation:*

$$\#\,(R^{(M,N)}_{m,n}[l_1,l_2,l_3])=\sum_{\substack{0\leq a\leq l_3\\ 0\leq c\leq l_2-a}}\#\,(R^{(M,N-1)}_{m-a,n-a-c}[l_1',l_2',l_3']),$$

where

$$\begin{aligned} l_1' &= l_1 + c - a - (l_1 + c - k)^+, \\ l_2' &= k - c, \\ l_3' &= l_1' + l_2' - k. \end{aligned} \tag{2.16}$$

In what follows, we fix the notation l_1', l_2', l_3' to be the integers given by (2.16), and $b = a + c$. Theorem 2.2.1 is proved in Sections 4, 5 and 6.

Let us outline the idea of the proof. We construct an explicit bijection

$$\mathfrak{m} : \bigsqcup_{\substack{0 \le a \le l_3 \\ 0 \le c \le l_2 - a}} R^{(M,N-1)}_{m-a,n-a-c}[l_1', l_2', l_3'] \to R^{(M,N)}_{m,n}[l_1, l_2, l_3]$$

in several steps. In Section 4.2, for $I, J \subset \{1, \dots, k\}$, we define the subsets $R^{(M,N)}_{m,n}[l_1, l_2]_{I,J} \subset R_{m,n}$ (the lower subsets). In Section 4.3, we define $R^{(M,N-1)}_{m-a,n-b}[l_1]^{I,J} \subset R_{m-a,n-b}$ (the upper subsets), where $a = \#(I)$ and $b = \#(J)$. In Section 4.4, we construct the bijection

$$\mathfrak{m}_{I,J} : R^{(M,N-1)}_{m-a,n-b}[l_1]^{I,J} \to R^{(M,N)}_{m,n}[l_1, l_2]_{I,J}.$$

In Section 5 we will prove that for each (l_1, l_2, l_3) satisfying (2.3)

$$R^{(M,N)}_{m,n}[l_1, l_2, l_3] = \bigsqcup_{\substack{I,J \subset \{1,\dots,k\} \\ \#(I) \le l_3, \ \#(J) \le l_2}} R^{(M,N)}_{m,n}[l_1, l_2]_{I,J},$$

and in Section 6 that for each (l_1, a, c) and (l_1', l_2', l_3') determined by (2.16)

$$R^{(M,N-1)}_{m-a,n-b}[l_1', l_2', l_3'] = \bigsqcup_{\substack{I,J \subset \{1,\dots,k\} \\ \#(I) = a, \#(J) = b}} R^{(M,N-1)}_{m-a,n-b}[l_1]^{I,J}.$$

This will complete the proof of Theorem 2.2.1.

An important implication of Theorem 2.2.1, and the main interest we have in proving it, is the following result.

Corollary 2.2.2. *Fix an integer $k \in \mathbb{Z}_{\ge 1}$, and consider the spaces of coinvariants of the $\widetilde{\mathfrak{H}}$-modules, $W^{(M,N)}_k[l_1, l_2, l_3]$ (see (3.7),(3.8)) and the sets of rigged partitions*

$$R^{(M,N)}[l_1, l_2, l_3] = \sqcup_{m,n} R^{(M,N)}_{m,n}[l_1, l_2, l_3].$$

Then

$$\dim W^{(M,N)}_k[l_1, l_2, l_3] = \#(R^{(M,N)}[l_1, l_2, l_3]). \tag{2.17}$$

Proof. Using Theorem 1.2.1 with $z_1 = z_2 = q = 1$, we see that these two sets of numbers satisfy the same recursion with the same initial condition. ☐

3. Functional realization of dual spaces and character formulas

In this section we identify the space dual to the module $W_k[l_1, l_2, l_3]_{m,n}$ with a certain space of rational functions in $m+n$ variables. We introduce a filtration in this space and describe the adjoint graded space explicitly by using the rigged partitions. As a corollary we compute the character of the space of coinvariants $W^{(M,N)}_k[l_1, l_2, l_3]$.

3.1. **Dual of the universal enveloping algebra.** Let $\widetilde{\mathfrak{H}}$ be the Heisenberg loop algebra with generators e_i, f_i, h_i $(i \in \mathbb{Z})$ and relations

$$[e_i, f_j] = h_{i+j}, \qquad [e_i, h_j] = [f_i, h_j] = 0.$$

Consider its universal enveloping algebra $U\widetilde{\mathfrak{H}}$. The algebra $U\widetilde{\mathfrak{H}}$ is graded by

$$\deg e_i = (1,0), \quad \deg f_i = (0,1), \quad \deg h_i = (1,1).$$

Let $(U\widetilde{\mathfrak{H}})_{m,n}$ be the subspace of degree (m,n). We construct the space dual to $(U\widetilde{\mathfrak{H}})_{m,n}$ in the space of rational functions in the variables $(x_1, \dots, x_m; y_1, \dots, y_n)$.

Consider the space of rational functions

$$\begin{aligned} \mathcal{F}_{m,n} \quad = \quad & \{F = \frac{p}{\prod_{i,j}(x_i - y_j)} \ : \ p \in \mathbb{C}[x_1^{\pm 1}, \dots, x_m^{\pm 1}, y_1^{\pm 1}, \dots, y_n^{\pm 1}], \\ & \text{symmetric in } x_1, \dots, x_m \text{ and } y_1, \dots, y_n \text{ separately,} \\ & \text{where } p = 0 \text{ if } x_1 = x_2 = y_1 \text{ or } x_1 = y_1 = y_2.\} \end{aligned} \tag{3.1}$$

There exists a coupling between $(U\widetilde{\mathfrak{H}})_{m,n}$ and $\mathcal{F}_{m,n}$. In order to define it, consider the mappings $L_{e_i} : \mathcal{F}_{m,n} \to \mathcal{F}_{m-1,n}$, $L_{f_i} : \mathcal{F}_{m,n} \to \mathcal{F}_{m,n-1}$, $L_{h_i} : \mathcal{F}_{m,n} \to \mathcal{F}_{m-1,n-1}$:

$$L_{e_i}(F) = \oint \frac{dx_1}{2\pi\sqrt{-1}x_1} F x_1^{-i}, \tag{3.2}$$

$$L_{f_i}(F) = \oint \frac{dy_1}{2\pi\sqrt{-1}y_1} F y_1^{-i}, \tag{3.3}$$

$$L_{h_i}(F) = \oint \frac{dy_1}{2\pi\sqrt{-1}y_1} \Big\{(x_1 - y_1)F\Big\}\Big|_{x_1 = y_1} y_1^{-1-i}, \tag{3.4}$$

where $F \in \mathcal{F}_{m,n}$. In each of these equations, we take the contour of integration to be a circle in $\mathbb{C}$ oriented counter-clockwise such that all the poles are inside. Because of the vanishing of p at $x_1 = x_2 = y_1$ and $x_1 = y_1 = y_2$, the integrand of (3.4) has the only pole in y_1 at $y_1 = 0$.

Similarly, we define the mappings R_{e_i}, R_{f_i}, R_{h_i} by the same formulas (3.2), (3.3), (3.4), respectively, using a contour such that all the poles except the origin are outside. As we noted above, we have $L_{h_i} = R_{h_i}$.

The following proposition is standard. We omit the proof.

Proposition 3.1.1. *There exists a unique coupling between $(U\widetilde{\mathfrak{H}})_{m,n}$ and $\mathcal{F}_{m,n}$ such that*

$$\begin{aligned} & \langle 1, 1 \rangle = 1 \\ & \langle e_i w, F \rangle = \langle w, L_{e_i}(F) \rangle, \quad \langle f_i w, F \rangle = \langle w, L_{f_i}(F) \rangle, \quad \langle h_i w, F \rangle = \langle w, L_{h_i}(F) \rangle, \\ & \langle w e_i, F \rangle = \langle w, R_{e_i}(F) \rangle, \quad \langle w f_i, F \rangle = \langle w, R_{f_i}(F) \rangle, \quad \langle w h_i, F \rangle = \langle w, R_{h_i}(F) \rangle. \end{aligned}$$

For example, it follows immediately that

Lemma 3.1.2. *If $w = e_{i_1} \dots e_{i_m} f_{j_1} \dots f_{j_n}$, then the coupling $\langle w, F \rangle$ is equal to the coefficient of $x_1^{i_1} \dots x_m^{i_m} y_1^{j_1} \dots y_n^{j_n}$ in the Laurent series obtained by expanding F in positive powers of y_j / x_i.*

Proposition 3.1.3. *The coupling given by Proposition 3.1.1 is non-degenerate.*

Proof. First we show that for any nonzero $F \in \mathcal{F}_{m,n}$ there exists $w \in (U\widetilde{\mathfrak{H}})_{m,n}$ such that $\langle w, F\rangle \neq 0$.

Consider the lexicographic ordering of monomials $x_1^{i_1}\dots x_m^{i_m}y_1^{j_1}\dots y_n^{j_n}$. Namely, the monomial $x_1^{i_1}\dots x_m^{i_m}y_1^{j_1}\dots y_n^{j_n}$ is higher than $x_1^{i'_1}\dots x_m^{i'_m}y_1^{j'_1}\dots y_n^{j'_n}$ if $i_1 > i'_1$, or if $i_1 = i'_1$ and $i_2 > i'_2$, and so on. Let $x_1^{i_1}\dots x_m^{i_m}y_1^{j_1}\dots y_n^{j_n}$ be the highest monomial present in p of F in (3.1). Then, taking $w = e_{i_1-n}\dots e_{i_m-n}f_{j_1}\dots f_{j_n}$ we have $\langle w, F\rangle \neq 0$.

Next we show that for any nonzero $w \in (U\widetilde{\mathfrak{H}})_{m,n}$ there exists $F \in \mathcal{F}_{m,n}$ such that $\langle w, F\rangle \neq 0$. For $l \le \min(m,n)$ let Z_l be the set of indices $(\mathbf{k},\mathbf{i},\mathbf{j})$ such that $\mathbf{k}\in\mathbb{Z}^l$, $\mathbf{i}\in\mathbb{Z}^{m-l}$ and $\mathbf{j}\in\mathbb{Z}^{n-l}$, with

$$k_1 \le \ldots \le k_l,\ i_1 \le \ldots \le i_{m-l},\ j_1 \le \ldots \le j_{n-l}.$$

By the PBW theorem the monomials

$$M[\mathbf{k},\mathbf{i},\mathbf{j}] := h_{k_1}\dots h_{k_l} e_{i_1}\dots e_{i_{m-l}} f_{j_1}\dots f_{j_{n-l}} \tag{3.5}$$

span $(U\widetilde{\mathfrak{H}})_{m,n}$. Set

$$F[\mathbf{k},\mathbf{i},\mathbf{j}] = \mathrm{Sym}\Big(\prod_{a=1}^{l}\frac{y_a^{k_a}}{x_a-y_a}\prod_{b=1}^{m-l}x_{l+b}^{i_b}\prod_{c=1}^{n-l}y_{l+c}^{j_c}\Big).$$

Take $(\mathbf{k},\mathbf{i},\mathbf{j})\in Z_l$ and $(\mathbf{k}',\mathbf{i}',\mathbf{j}')\in Z_{l'}$. Using the definition of the coupling, we have

$$\langle M[\mathbf{k},\mathbf{i},\mathbf{j}], F[\mathbf{k}',\mathbf{i}',\mathbf{j}']\rangle = \begin{cases} 0 & \text{if } l > l';\\ \delta_{\mathbf{k},\mathbf{k}'}\delta_{\mathbf{i},\mathbf{i}'}\delta_{\mathbf{j},\mathbf{j}'} & \text{if } l = l'.\end{cases} \tag{3.6}$$

The assertion follows from this. ☐

3.2. **Dual to the $\widetilde{\mathfrak{H}}$-module and coinvariant.** Following [FKLMM2], define the $\widetilde{\mathfrak{H}}$-module $W[l_1,l_2,l_3]$ as a quotient of $U\widetilde{\mathfrak{H}}$ by the left ideal generated by the elements

$$x_i\ (i\le 0, x\in\mathfrak{H}),\ e_1^{l_1+1},\quad f_1^{l_2+1},\quad h_1^{l_3+1}, \tag{3.7}$$

and the level-k restricted module $W_k[l_1,l_2,l_3]$ is the quotient of $W[l_1,l_2,l_3]$ by the two-sided ideal generated by

$$e(z)^{k+1},\quad f(z)^{k+1}, \tag{3.8}$$

where we used the generating series $e(z) = \sum_{i\in\mathbb{Z}} e_i z^i$, $f(z) = \sum_{i\in\mathbb{Z}} f_i z^i$. (Strictly speaking, these elements are in the completion of $U\widetilde{\mathfrak{H}}$; however as usual, the module is in the category $\mathcal{O}$ due to (3.7) and, when acting in $W[l_1,l_2,l_3]$, they are finite sums in $U\widetilde{\mathfrak{H}}$.)

The dual space of $W_k[l_1,l_2,l_3]_{m,n}$ is realized in $\mathcal{F}_{m,n}$ as the subspace orthogonal to these ideals. We denote this subspace by $W_k[l_1,l_2,l_3]^*_{m,n}$. The following theorem is a consequence of Proposition 3.1.1

Theorem 3.2.1. *The space* $W_k[l_1,l_2,l_3]^*_{m,n}$ *is given by*

$$\begin{aligned}W_k[l_1,l_2,l_3]^*_{m,n} \quad = \quad & \{F=\frac{\prod_i x_i\prod_j y_j}{\prod_{i,j}(x_i-y_j)}f\in\mathfrak{F}_{m,n};\\ & f\in\mathbb{C}[x_1,...,x_m,y_1,...,y_n]\ ,\\ & f=0\ \textit{if}\ x_1=\cdots=x_{k+1}\ \textit{or}\ y_1=\cdots=y_{k+1}\textit{or}\\ & x_1=\cdots=x_{l_1+1}=0\ \textit{or}\ y_1=\cdots=y_{l_2+1}=0,\\ & \prod_{i=1}^{l_3}\left(\frac{\partial}{\partial x_{i+1}}\frac{\partial}{\partial y_{i+1}}\right)^i f=0\ \textit{if}\\ & x_1=\cdots=x_{l_3+1}=y_1=\cdots=y_{l_3+1}=0.\}\end{aligned}$$

Take $M,N\in\mathbb{Z}_{\geq 0}$. Let the subalgebra $\mathfrak{a}^{(M,N)}$ of $\widetilde{\mathfrak{H}}$ be generated by the elements e_i $(i\geq M)$ and f_i $(i\geq N)$. Following [FKLMM2], define the space of coinvariants by

$$W_k^{(M,N)}[l_1,l_2,l_3]=\oplus_{m,n}W_k^{(M,N)}[l_1,l_2,l_3]_{m,n}, \tag{3.9}$$

$$W_k^{(M,N)}[l_1,l_2,l_3]_{m,n}=\begin{cases}0 & \text{if } M=0 \text{ and } n-2m<k-l_1;\\ 0 & \text{if } N=0 \text{ and } m-2n<k-l_2;\\ W_k[l_1,l_2,l_3]/\mathfrak{a}^{(M,N)}W_k[l_1,l_2,l_3], & \text{otherwise.}\end{cases}$$

Define $\mathfrak{F}^{(M,N)}_{m,n}\subset\mathfrak{F}_{m,n}$ to be the subset consisiting of functions F satisfying the degree restrictions

$$\deg_{x_1}F<M,\quad \deg_{y_1}F<N. \tag{3.10}$$

Here the degree of the rational function F in the variable x_1 is defined to be the highest power in x_1 appearing in the Laurent series expansion of F in positive powers of y_j/x_1. In other words, we have $\deg_{x_1}F=1-n+\deg_{x_1}f$. Similarly, we have $\deg_{y_1}F=1-m+\deg_{y_1}f$. If m or n is zero, the corresponding degree restriction is void.

Definition 3.2.2. We define the space of rational functions $W_k^{(M,N)}[l_1,l_2,l_3]^*$ by

$$\begin{aligned}W_k^{(M,N)}[l_1,l_2,l_3]^* \quad &= \quad \oplus_{m,n}W_k^{(M,N)}[l_1,l_2,l_3]^*_{m,n},\\ W_k^{(M,N)}[l_1,l_2,l_3]^*_{m,n} \quad &= \quad \begin{cases}0 & \text{if } M=0 \text{ and } n-2m<k-l_1;\\ 0 & \text{if } N=0 \text{ and } m-2n<k-l_2;\\ W_k[l_1,l_2,l_3]^*_{m,n}\cap\mathfrak{F}^{(M,N)}_{m.n} & \text{otherwise.}\end{cases}\end{aligned} \tag{3.11}$$

The space $W_k^{(M,N)}[l_1,l_2,l_3]^*_{m,n}$ is finite-dimensional and dual to $W_k^{(M,N)}[l_1,l_2,l_3]_{m,n}$ given by (3.9).

3.3. **Polynomials with Serre relations.** In this section we study symmetric polynomials of the form $f(x_1,\dots,x_m;y_1,\dots,y_n)$ which vanish when $x_1=x_2=y_1$ or $x_1=y_1=y_2$. Proposition 3.3.3 will be used in the proof of Theorem 3.5.7 (see Lemma 3.5.4).

For a function $f(x_1,\dots,x_m;y_1,\dots,y_n)$ and $a_i,b_j\in\mathbb{Z}_{\geq 0}$ denote

$$[a_1,\dots,a_s;b_1,\dots,b_t]f:=\prod_{i=1}^{s}\Big(\frac{\partial}{\partial x_i}\Big)^{a_i}\prod_{i=1}^{t}\Big(\frac{\partial}{\partial y_i}\Big)^{b_i}f|_{x_1=\cdots=x_s=y_1=\cdots=y_t=z}.$$

We also denote the functions

$$\Big(\frac{\partial}{\partial z}\Big)^r\prod_{i=3}^{s}\Big(\frac{\partial}{\partial x_i}\Big)^{a_i}\prod_{i=2}^{t}\Big(\frac{\partial}{\partial y_i}\Big)^{b_i} f(z,z,x_3,\dots;z,y_2,\dots)|_{x_3=\dots=x_s=y_2=\dots=y_t=z},$$

$$\Big(\frac{\partial}{\partial z}\Big)^r\prod_{i=2}^{s}\Big(\frac{\partial}{\partial x_i}\Big)^{a_i}\prod_{i=3}^{t}\Big(\frac{\partial}{\partial y_i}\Big)^{b_i} f(z,x_2,\dots;z,z,y_3,\dots)|_{x_2=\dots=x_s=y_3=\dots=y_t=z}$$

by

$$[a_3,\dots,a_s,(*,*;*)^r,b_2,\dots,b_t]f,\qquad [a_2,\dots,a_s,(*;*,*)^r,b_3,\dots,b_t]f.$$

If the number of x variables in f is smaller than the s or the number of y variables is smaller than t then we define the functions $[a_1,\dots,a_s;b_1,\dots,b_t]f$, $[a_3,\dots,a_s,(*,*;*)^r,b_2,\dots,b_t]f$, $[a_2,\dots,a_s,(*;*,*)^r,b_3,\dots,b_t]f$ to be 0.

We have relations

$$[a_3,\dots,a_s,(*,*;*)^r,b_2,\dots,b_t]f=\sum_{a_1+a_2+b_1=r}\frac{r!}{a_1!a_2!b_1!}[a_1,\dots,a_s;b_1,\dots,b_t]f, \tag{3.12}$$

$$[a_2,\dots,a_s,(*;*,*)^r,b_3,\dots,b_t]f=\sum_{a_1+b_1+b_2=r}\frac{r!}{a_1!b_1!b_2!}[a_1,\dots,a_s;b_1,\dots,b_t]f. \tag{3.13}$$

For the rest of this section, let $f(x,y)$ be a polynomial in $x_1,\dots,x_m,y_1,\dots,y_n$, symmetric with respect to permutations of x and to permutations of y, satisfying the *Serre* relations:

$$f(x,y)=0\qquad\text{if}\ \ x_1=x_2=y_1\quad\text{or}\ \ x_1=y_1=y_2. \tag{3.14}$$

Note that now $[a_1,\dots,a_s;b_1,\dots,b_t]f$ does not depend on the order of a_i or b_j. Also we have

$$[a_3,\dots,a_s,(*,*;*)^r,b_2,\dots,b_t]f=[a_2,\dots,a_s,(*;*,*)^r,b_3,\dots,b_t]f=0.$$

In particular we have many linear relations among $[a_1,\dots,a_s;b_1,\dots,b_t]f$ thanks to (3.12), (3.13). The following lemma describes some of the relations which consist of a single term.

Lemma 3.3.1. *For $s,t\in\mathbb{Z}_{\geq 0}$, we have the identities*

$$[0,1,2,\dots,s,s,s^t;0,1,2,\dots,s-1,s+t]f=0, \tag{3.15}$$

$$[0,1,2,\dots,s-1,s+t;0,1,2,\dots,s,s,s^t]f=0. \tag{3.16}$$

where s^t denotes $s,s\dots,s$ repeated t times.

Proof. We use induction on s.

The identity (3.15) for $s=0$ takes the form

$$[0^{t+2};t]f=0,\qquad t\in\mathbb{Z}_{\geq 0}, \tag{3.17}$$

The case $t=0$ is just the Serre relation: $[0,0;0]f=0$. We obtain (3.17) by induction on t. Assume $[0^{t+2};t]f=0$, $t=0,\dots,t_0-1$. Then $[0^{t_0+2};t_0]f=0$ follows from the identity $[0^{t_0}(*,*;*)^{t_0}]f=0$. Indeed on the RHS of (3.12) for $[0^{t_0}(*,*;*)^{t_0}]$, the only term left is exactly $[0^{t_0+2};s_0]f=0$. The $s=0$ case of identity (3.16) is proved similarly.

Now assume (3.15), (3.15) are proved for $s=0,\dots,s_0-1$ and let us prove them for $s=s_0$. It is enough to prove (3.15), then (3.16) is done by the same argument switching the roles of x and y.

We use induction on t. The case $t=0$ follows from the identity

$$[0,1,2,\dots,s_0-1,(*,*;*)^{3s_0},0,1,2,\dots,s_0-1]f=0.$$

Suppose we have the statement for $t = 0, \dots, t_0 - 1$, then the case $t = t_0$ follows from the identity

$$[0, \dots, s_0 - 1, s_0^{t_0}, (*, *; *)^{3s_0+t_0}, 0, 1, \dots, s_0 - 1]f = 0.$$

□

Now we derive more identities under additional assumptions.

For a function $g(x_1, \dots, x_m; y_1, \dots, y_n)$ and $a_i, b_j \in \mathbb{Z}_{\geq 0}$, we denote

$$\begin{aligned}
&[a_1, \dots, a_s; b_1, \dots, b_t]' g = [a_1, \dots, a_s; b_1, \dots, b_t] g|_{z=0} \\
&[a_3, \dots, a_s, (*, *; *)^r, b_2, \dots, b_t]' g = [a_3, \dots, a_s, (*, *; *)^r, b_2, \dots, b_t] g|_{z=0}, \\
&[a_2, \dots, a_s, (*; *, *)^r, b_3, \dots, b_t]' g = [a_2, \dots, a_s, (*; *, *)^r, b_3, \dots, b_t] g|_{z=0}.
\end{aligned}$$

Then the $h_1^{l_3+1} = 0$ relation is translated into

$$[0, 1, \dots, l_3; 0, 1, \dots, l_3]' f = 0, \tag{3.18}$$

see Theorem 3.2.1.

Remark 3.3.2. The condition $e_1^{l_1+1} = 0$ reads $[0^{l_1+1}; \emptyset]' f = 0$. It follows from our results that we automatically have $h_1^{l_1+1} = 0$. It is an instructive exercise to prove $[0, 1, \dots, l_1; 0, 1, \dots, l_1]' f = 0$ starting from $[0^{l_1+1}; \emptyset]' f = 0$ and using (3.12), (3.13).

Proposition 3.3.3. *Let f satisfy (3.18). Then for $s \in \mathbb{Z}_{\geq l_3}$ we have the identity*

$$[0^{s+1}; s^{l_3+1}]' f = 0. \tag{3.19}$$

Proof. We use the induction on s. Assume the statement is proved for $s = l_3, \dots, s_0 - 1$. (We assume nothing if $s_0 = l_3$.) We will prove it for $s = s_0$. To do that we prove by the inverse induction on r the identity

$$[0, 1, \dots, r - 1, (r)^{s_0 - r + 1}; 0, 1 \dots, r - 1, (s_0)^{l_3 - r + 1}]' f = 0, \tag{3.20}$$

where $r = l_3 + 1, l_3, l_3 - 1, \dots, 0$. The case $r = 0$ is exactly (3.19) for $s = s_0$.

The identity (3.20) for $r = l_3 + 1$ follows directly from (3.18). Assume we have (3.20) for $r = l_3 + 1, l_3, \dots, r_0 + 1$. Let us prove it for $r = r_0$. For that we prove the identity

$$[0, 1, \dots, r_0, (r_0)^q (r_0 + 1)^{s_0 - r_0 - q}; 0, 1 \dots, r_0 - 1, r_0 + q, (s_0)^{l_3 - r_0}]' f = 0,$$

for $q = 0, \dots, s_0 - r_0$ by induction on q. For $q = 0$ we have exactly (3.20) for $r = r_0 + 1$. If the statement is proved for $q = 0, \dots, q_0 - 1$ then the statement for $q = q_0$ follows from the relation

$$[0, 1, \dots, r_0 - 1, (r_0)^{q_0 - 1} (r_0 + 1)^{s_0 - r_0 - q_0} (**; *)^{3r_0 + q_0}, 0, 1 \dots, r_0 - 1, (s_0)^{l_3 - r_0}]' f = 0$$

and (3.15).

For $q = s_0 - r_0$ we obtain (3.20) for $r = r_0$ and the proof is finished. □

3.4. **Multiplication of functional spaces.** In this section we describe a multiplicative structure which relates the functional spaces for different levels k. Though the results of this section are not used in what follows, we think that Theorem 3.4.2, is interesting in its own right.

Fix $k^{(j)}$, $l_i^{(j)}$ $(i = 1, 2, 3,\ j = 1, 2)$ and set $k = k^{(1)} + k^{(2)}$, $l_i = l_i^{(1)} + l_i^{(2)}$ $(i = 1, 2, 3)$.

Let $\Delta :\ U\widetilde{\mathfrak{H}} \to U\widetilde{\mathfrak{H}} \otimes U\widetilde{\mathfrak{H}}$ be the usual comultiplication defined by the rule $\Delta(g) = 1 \otimes g + g \otimes 1$ for $g \in \widetilde{\mathfrak{H}}$. We also denote by Δ the map of $U\widetilde{\mathfrak{H}}$ modules

$$\Delta :\ W_k[l_1, l_2, l_3] \to W_{k^{(1)}}[l_1^{(1)}, l_2^{(1)}, l_3^{(1)}] \otimes W_{k^{(2)}}[l_1^{(2)}, l_2^{(2)}, l_3^{(2)}]$$

uniquely determined by the condition $\Delta(v) = v^{(1)} \otimes v^{(2)}$, where v, $v^{(1)}$ and $v^{(2)}$ are the highest weight vectors of the corresponding modules.

The map Δ descends to the spaces of coinvariants

$$\Delta^{(M,N)}\ W_k^{(M,N)}[l_1, l_2, l_3] \to W_{k^{(1)}}^{(M,N)}[l_1^{(1)}, l_2^{(1)}, l_3^{(1)}] \otimes W_{k^{(2)}}^{(M,N)}[l_1^{(2)}, l_2^{(2)}, l_3^{(2)}].$$

By Proposition 6.3.3 in [FKLMM2] the map $\Delta^{(M,N)}$ is injective.

Define the map

$$*:\ W^*_{k^{(1)}}[l_1^{(1)}, l_2^{(1)}, l_3^{(1)}] \otimes W^*_{k^{(2)}}[l_1^{(2)}, l_2^{(2)}, l_3^{(2)}] \to W^*_k[l_1, l_2, l_3]$$

by the following rule. Let $F^{(j)}(x_1^{(j)}, \dots, x_{m^{(j)}}^{(j)}; y_1^{(j)}, \dots, y_{n^{(j)}}^{(j)} \in W^*_{k^{(j)}}[l_1^{(j)}, l_2^{(j)}, l_3^{(j)}]_{m^{(j)},n^{(j)}}$, $(j = 1, 2)$. Then $F^{(1)} * F^{(2)} \in W^*_k[l_1, l_2, l_3]_{m,n}$, where $m = m^{(1)} + m^{(2)}$, $n = n^{(1)} + n^{(2)}$, is given by

$$\begin{aligned} &F^{(1)} * F^{(2)}(x_1, \dots, x_m; y_1, \dots, y_n) = \\ &\quad = \mathrm{Sym}\left(F^{(1)}(x_1, \dots, x_{m^{(1)}}; y_1, \dots, y_{n^{(1)}})\ F^{(2)}(x_{m^{(1)}+1}, \dots, x_m; y_{n^{(1)}+1}, \dots, y_n)\right). \end{aligned}$$

Here Sym denotes the symmetrization with respect to two groups of variables $x_1, \dots, x_m$ and $y_1, \dots, y_n$.

Lemma 3.4.1. *The map $*$ is well defined. Moreover, the map $*$ is dual to the map Δ:*

$$\langle \Delta(w), F^{(1)} \otimes F^{(2)} \rangle = \langle w, F^{(1)} * F^{(2)} \rangle, \tag{3.21}$$

where $w \in W_k[l_1, l_2, l_3]$ and the pairing on the tensor product of vector spaces is standard: $\langle v^{(1)} \otimes v^{(2)}, F^{(1)} \otimes F^{(2)} \rangle = \langle v^{(1)}, F^{(1)} \rangle \langle v^{(2)}, F^{(2)} \rangle$.

Proof. The fact that the map $*$ is well defined follows directly from the defintion. Note that the vectors w of the form $w = e_{i_1} \dots e_{i_m} f_{j_1} \dots f_{j_n} v$, where v is the highest weight vector, span $W_k[l_1, l_2, l_3]_{m,n}$. Indeed, as shown in the proof of Proposition 3.1.3, the orthogonal complement of the span of such vectors is trivial. Therefore it is enough to check (3.21) for w. For such vectors the equation (3.21) is clear from Lemma 3.1.2. ☐

The map $*$ obviously descends to the spaces dual to the coinvariants:

$$*^{(M,N)} :\ W_{k^{(1)}}^{*(M,N)}[l_1^{(1)}, l_2^{(1)}, l_3^{(1)}] \otimes W_{k^{(2)}}^{*(M,N)}[l_1^{(2)}, l_2^{(2)}, l_3^{(2)}] \to W_k^{*(M,N)}[l_1, l_2, l_3].$$

From Lemma 3.4.1 and the injectivity of the coproduct, Proposition 6.3.3 in [FKLMM2], we obtain

Theorem 3.4.2. *The map $*^{(M,N)}$ is surjective.*

This is a rather simple statement for certain spaces of symmetric functions. However, we do not know of any direct proof of this statement.

3.5. **Filtration of $W_k^*[l_1,l_2,l_3]$.** Let μ be a level-k restricted partition of m of the form (2.1). We will define a map φ_μ which sends functions of the variables $(x_1,\dots,x_m)$ to functions of the variables $\{x_j^{(\alpha)}\}_{\alpha\in I_k, 1\le j\le m_\alpha(\mu)}$.

Fix a numbering from 1 to m of the set of indices (α,j) where $\alpha\in I_k$ and $1\le j\le m_\alpha(\mu)$. We define $\varphi(x_i)=x_j^{(\alpha)}$ where (α,j) is the i-th index in this numbering. The μ-evaluation map φ_μ is defined by

$$\varphi_\mu\ \big(F(x_1,...,x_m)\big)\ = F(\varphi_\mu(x_1),...,\varphi_\mu(x_m)).$$

If F is a symmetric function, then $\varphi_\mu(F)$ is symmetric in the variables $(x_1^{(\alpha)},\dots,x_{m_\alpha}^{(\alpha)})$ with fixed α. Moreover, $\varphi_\mu(F)$ is independent of the choice of the numbering.

Given a pair of partitions (μ,ν) of (m,n), (μ,ν)-evaluation $\varphi_{\mu,\nu}$ is defined by

$$\varphi_{\mu,\nu}(F(x_1,\dots,x_m;y_1,\dots,y_n)) = F(\varphi_\mu(x_1),...,\varphi_\mu(x_m);\varphi_\nu(y_1),...,\varphi_\nu(y_n)).$$

Partitions are ordered lexicographically, $\mu>\mu'$ if and only if there exists some i for which $\mu_i>\mu'_i$ and $\mu_j=\mu'_j$ for all $j<i$. Similarly, pairs of partitions (μ,ν) are ordered, $(\mu,\nu)>(\mu',\nu')$ if and only if $\mu>\mu'$, or $\mu=\mu'$ and $\nu>\nu'$.

Now suppose $F\in W_k^*[l_1,l_2,l_3]$. Since F does not have a pole at $x_i=x_j$ or $y_i=y_j$, the (μ,ν)-evaluation is well-defined. Consider the subspaces

$$\begin{aligned}\operatorname{Ker}\varphi_{\mu,\nu} &\subset W_k^*[l_1,l_2,l_3],\\ \Gamma_{\mu,\nu} &= \cap_{(\mu',\nu')>(\mu,\nu)}\operatorname{Ker}\varphi_{\mu,\nu},\\ \Gamma'_{\mu,\nu} &= \Gamma_{\mu,\nu}\cap\operatorname{Ker}\varphi_{\mu,\nu}.\end{aligned}\tag{3.22}$$

The subspaces $\Gamma_{\mu,\nu}$ give a filtration of $W_k^*[l_1,l_2,l_3]$. Our goal is to characterize the adjoint graded space $\mathrm{Gr}_{\mu,\nu}=\Gamma_{\mu,\nu}/\Gamma'_{\mu,\nu}$ (see Theorem 3.5.7).

Lemma 3.5.1. *Let $F\in\Gamma_{\mu,\nu}$. The function $\varphi_{\mu,\nu}(F)$ has a zero of order at least $2\min(\alpha,\beta)$ if $x_i^{(\alpha)}=x_j^{(\beta)}$ or $y_i^{(\alpha)}=y_j^{(\beta)}$.*

Proof. Consider the case $x_i^{(\alpha)}=x_j^{(\beta)}$ with $\alpha\ge\beta$, with α,β fixed. Denote the variables x_k such that $\varphi_\mu(x_k)=x_j^{(\beta)}$ by $x_{j,l}^{(\beta)}$ $(l=1,\dots,m_\beta)$ in some ordering.

We can carry out the evaluation in two steps: $\varphi_{\mu,\nu}(F)=\varphi_2(\varphi_1(F))$, where φ_1 is the evaluation of all the variables except $x_{j,l}^{(\beta)}$ $(l=1,\dots,m_\beta)$ and φ_2 is the evaluation of the variables $x_{j,l}^{(\beta)}$. Let $F_1=\varphi_1(F)$. Since $\alpha\ge\beta$ and $F\in\Gamma_{\mu,\nu}$, we have

$$F_1\Big|_{x_{j,l}^{(\beta)}=x_i^{(\alpha)}}=0,\qquad 1\le l\le\beta.$$

Differentiating the left hand side of this equality by $x_i^{(\alpha)}$ and using the symmetry of F with respect to $(x_1,,\dots,x_m)$, we can deduce that

$$\frac{\partial F_1}{\partial x_{j,l}^{(\beta)}}\Bigg|_{x_{j,l}^{(\beta)}=x_i^{(\alpha)}}=0,\qquad 1\le l\le\beta.$$

Therefore, F_1 has a zero of order at least two at $x_{j,l}^{(\beta)}=x_i^{(\alpha)}$ for each l. After evaluation, $\varphi_2(F_1)$ is divisible by $(x_i^{(\alpha)}-x_j^{(\beta)})^{2\beta}$. $\square$

Lemma 3.5.2. *Let $F \in \Gamma_{\mu,\nu}$. The function $\varphi_{\mu,\nu}(F)$ has a pole of order at most* $\min(\alpha,\beta)$ *if* $x_i^{(\alpha)} = y_j^{(\beta)}$.

Proof. Without loss of generality, we can assume that $\alpha \geq \beta$. If $\alpha = 1$ the assertion follows immediately.

Suppose $\alpha \geq 2$. Set $g = \varphi_{\mu,\nu}(f)$, where f is the polynomial function of Theorem 3.2.1. It is enough to show that g is divisible by $(x_i^{(\alpha)} - y_j^{(\beta)})^{(\alpha-1)\beta}$, because the evaluation of the prefactor in Theorem 3.2.1 only contains a pole of order $\alpha\beta$ at this point.

Let $Y_{[j,\beta]} = \{y_{j,l}^{(\beta)}\}_{1\leq l\leq\beta} = \varphi_{\mu,\nu}^{-1}(y_i^{(\beta)})$. We obtain g in two steps: $\varphi_{\mu,\nu}(f) = \varphi_2(\varphi_1(f))$, where φ_1 is the evaluation of all the variables except those in $Y_{[j,\beta]}$.

Using the fact that $f = 0$ if $x_1 = x_2 = y_1$,

$$\frac{\partial^s}{\partial (x_i^{(\alpha)})^s}\, \varphi_1(f)\Big|_{y_{j,l}^{(\beta)} = x_i^{(\alpha)}} = 0, \qquad 0 \leq s \leq \alpha - 2, \quad 1 \leq l \leq \beta.$$

Therefore, $\varphi_1(f)$ is divisible by $(y_{j,l}^{(\beta)} - x_i^{(\alpha)})^{\alpha-1}$, and hence g is divisible by $(x_i^{(\alpha)} - y_j^{(\beta)})^{(\alpha-1)\beta}$. □

Lemma 3.5.3. *Let $F \in \Gamma_{\mu,\nu}$ and f be as in Theorem 3.2.1. The function $\varphi_{\mu,\nu}(f)$ has a zero of order at least $(\alpha - l_1)^+$ (resp., $(\alpha - l_2)^+$) if $x_i^{(\alpha)} = 0$ (resp., $y_i^{(\alpha)} = 0$).*

Proof. The assertion follows by a similar argument as in the proof of Lemma 3.5.2 from the restriction on f that it is zero if $x_1 = \cdots = x_{l_1+1} = 0$ or $y_1 = \cdots = y_{l_2+1} = 0$. □

Let $f(x,y)$ be a polynomial in two variables x and y. We say that f has a zero of order s at $x = y = 0$ if $f(tx,ty)$ has a zero of order s at $t = 0$.

Lemma 3.5.4. *Let $F \in \Gamma_{\mu,\nu}$ and f be as in Theorem 3.2.1. Then the function $\varphi_{\mu,\nu}(f)$ has a zero of order at least $\alpha\beta - l_3$ at $x_i^{(\alpha)} = y_j^{(\beta)} = 0$.*

Proof. If $l_3 \geq \min(\alpha,\beta)$ then there is nothing to prove due to Lemma 3.5.2. Therefore, without loss of generality we assume $l_3 + 1 \leq \alpha \leq \beta$. Let

$$h := f(x_1,\ldots,x_\alpha;\underbrace{y,\ldots,y}_{\beta}), \qquad g := \left(\prod_{i=1}^{\alpha}(x_i - y)^{\beta-1}\right)^{-1} h.$$

Note that for $i = 1,\ldots,\alpha$, we have

$$\left(\frac{\partial}{\partial y}\right)^s h|_{x_i=y} = 0, \qquad s = 0,\ldots,\beta-2,$$

because $f = 0$ if $x_i = y_j = y_t$. Therefore, g is a polynomial.

From Proposition 3.3.3 and (3.17) we obtain

$$\left(\frac{\partial}{\partial y}\right)^{s\beta - l_3 - 1} h|_{x_1=\cdots=x_s=y=0} = 0, \qquad s = l_3+1, l_3+2,\ldots,\alpha-1. \tag{3.23}$$

Now, it follows by induction on r that for $r = 0,\ldots,\alpha - l_3$ the polynomial g is of the form

$$g = y^r g_r' + \sum_{i=0}^{r-1} y^i g_i,$$

where g'_r is a polynomial and g_i are polynomials independent on y of degree at least $\alpha - l_3 - i$ in $x_1, \dots, x_\alpha$. Indeed, if we have the statement for $r = r_0 - 1$, then the case $r = r_0$ follows from (3.23) with $s = r_0 + l_3$.

Therefore g is of degree at least $\alpha - l_3$ in $x_1, \dots, x_\alpha, y$ and the lemma follows. ☐

Let μ and ν be level k partitions of m and n, respectively. Set

$$G_{\mu,\nu} = \prod_{\alpha,i}\Big(x_i^{(\alpha)}\Big)^{\alpha+(\alpha-l_1)^+}\prod_{\alpha,i}\Big(y_i^{(\alpha)}\Big)^{\alpha+(\alpha-l_2)^+}\frac{\prod(x_i^{(\alpha)}-x_j^{(\beta)})^{2\min(\alpha,\beta)}\prod(y_i^{(\alpha)}-y_j^{(\beta)})^{2\min(\alpha,\beta)}}{\prod(x_i^{(\alpha)}-y_j^{(\beta)})^{\min(\alpha,\beta)}}.$$

Consider the space of rational functions in the variables $\{x_i^{(\alpha)}\}$, $\{y_i^{(\alpha)}\}$ defined as follows:

$$\begin{aligned}\mathcal{G}_{\mu,\nu}[l_1,l_2,l_3] \quad = \quad & \{G = G_{\mu,\nu}g\ ;\ g \in \mathbb{C}[\{x_i^{(\alpha)}\},\{y_i^{(\alpha)}\}],\\ & g \text{ is invariant by the transposition } x_i^{(\alpha)} \leftrightarrow x_j^{(\alpha)} \text{ or } y_i^{(\alpha)} \leftrightarrow y_j^{(\alpha)},\\ & g \text{ has a zero at } x_i^{(\alpha)} = y_j^{(\beta)} = 0 \text{ of order at least } \tau^{(\alpha,\beta)}[l_1,l_2,l_3] \text{ given by (2.4).}\}\end{aligned} \tag{3.24}$$

We define the total homogeneous degree of G as the homogeneous degree of G in all the variables $x_i^{(\alpha)}$ and $y_i^{(\alpha)}$.

Proposition 3.5.5. *The evaluation map*

$$\varphi_{\mu,\nu} : W_k^*[l_1,l_2,l_3] \to \mathcal{G}_{\mu,\nu}[l_1,l_2,l_3], \quad F \mapsto \varphi_{\mu,\nu}(F)$$

is well-defined, injective and preserves the total homogeneous degree.

Proof. This follows from Lemmas 3.5.1, 3.5.2, 3.5.3 and 3.5.4. ☐

Lemma 3.5.6. *If F belongs to the subspace $W_k^{(M,N)}[l_1,l_2,l_3]^*_{m,n} \cap \Gamma_{\mu,\nu}$ defined by the conditions (3.10) and (3.22) then the function g given by $\varphi_{\mu,\nu}(F) = G_{\mu,\nu}g$ satisfies the degree restrictions*

$$\deg_{x_i^{(\alpha)}} g \le P_{\mu,\nu}^{(M)}[l_1]_\alpha, \quad \deg_{y_i^{(\alpha)}} g \le Q_{\mu,\nu}^{(N)}[l_2]_\alpha, \tag{3.25}$$

where $P_{\mu,\nu}^{(M)}[l_1], Q_{\mu,\nu}^{(N)}[l_2]$ are given by (2.6), (2.7).

The proof is straightforward.

Set

$$\begin{aligned}\mathcal{G}_k^{(M,N)}[l_1,l_2,l_3] \quad &= \quad \oplus_{m,n}\mathcal{G}_{m,n}^{(M,N)}[l_1,l_2,l_3],\\ \mathcal{G}_{m,n}^{(M,N)}[l_1,l_2,l_3] \quad &= \quad \{G = G_{\mu,\nu}g \in \mathcal{G}_{\mu,\nu}[l_1,l_2,l_3];\\ & \qquad |\mu| = m, \quad |\nu| = n,\\ & \qquad P_{\mu,\nu}^{(M)}[l_1] \ge 0, \quad Q_{\mu,\nu}^{(N)}[l_2] \ge 0,\\ & \qquad \deg_{x_i^{(\alpha)}} g \le P_{\mu,\nu}^{(M)}[l_1]_\alpha, \quad \deg_{y_i^{(\alpha)}} g \le Q_{\mu,\nu}^{(N)}[l_2]_\alpha.\}\end{aligned} \tag{3.26}$$

Consider the filtration of $W_k^{(M,N)}[l_1,l_2,l_3]^*_{m,n}$ consisting of the subspaces $W_k^{(M,N)}[l_1,l_2,l_3]^*_{m,n} \cap \Gamma_{\mu,\nu}$, and the adjoint graded space $\mathrm{Gr}(W_k^{*(M,N)}[l_1,l_2,l_3]_{m,n})$. We set

$$\mathrm{Gr}(W_k^{*(M,N)}[l_1,l_2,l_3]) = \oplus_{m,n}\mathrm{Gr}(W_k^{*(M,N)}[l_1,l_2,l_3]_{m,n})$$

The mappings $\varphi_{\mu,\nu}$ induce an injective map

$$\varphi : \mathrm{Gr}(W_k^{*(M,N)}[l_1,l_2,l_3]) \to \mathcal{G}_k^{(M,N)}[l_1,l_2,l_3] \tag{3.27}$$

In fact, this map is an isomorphism, however we do not know of a straightforward proof of the surjectivity. Nevertheless we can prove the following theorem:

Theorem 3.5.7. *The mapping φ of (3.27) is an isomorphism preserving the total homogeneous degree.*

Proof. For a rigged partition (μ,r) we denote by $m_r(\{x_i^{(\alpha)}\})$ the monomial symmetric polynomial corresponding to the monomial $\prod_{\alpha,i}\left(x_i^{(\alpha)}\right)^{r_i^{(\alpha)}}$. The space of rational functions $\mathcal{G}_{m,n}^{(M,N)}[l_1,l_2,l_3]$ can be parameterized by the set of rigged partitions $R_{m,n}^{(M,N)}[l_1,l_2,l_3]$ by associating $G_{\mu,\nu}g$ to $(\mu,r;\nu,s)$ where $g = m_r(\{x_i^{(\alpha)}\})m_s(\{y_i^{(\alpha)}\})$. The statement follows from the injectivity of φ and the equality of dimensions (2.17). □

3.6. **Characters of coinvariants.** The purpose of this section is to compute the characters of $W_k^{(M,N)}[l_1,l_2,l_3]$ and the space of $\widehat{\mathfrak{sl}}_2$-coinvariants.

The algebra $U\widetilde{\mathfrak{H}}$ has a triple grading given by (1.3). The spaces $W_k^{(M,N)}[l_1,l_2,l_3]$ are quotients of $U\widetilde{\mathfrak{H}}$ and have an induced grading on them. Define the characters of $W_k^{(M,N)}[l_1,l_2,l_3]$ to be

$$\mathrm{ch}_{\,z_1,z_2,q}W_k^{(M,N)}[l_1,l_2,l_3] = \sum_{m,n,d}\dim(W_k^{(M,N)}[l_1,l_2,l_3]_{m,n,d})\ z_1^m z_2^n q^d,$$

where $W_k^{(M,N)}[l_1,l_2,l_3]_{m,n,d}$ is the subspace of degree (m,n,d).

Note that the dual space $W_k^{(M,N)}[l_1,l_2,l_3]^*$ is similarly graded, with

$$\deg x_i = (1,0,1), \quad \deg y_i = (0,1,1).$$

Hence we can define the character of the function spaces described above. These are equal to those of the corresponding quotients of $U\widetilde{\mathfrak{H}}$.

The evaluation mapping preserves the degree. Hence, in the image of the evaluation by φ, the induced degree is

$$\deg x_i^{(\alpha)} = (1,0,1), \quad \deg y_i^{(\alpha)} = (0,1,1).$$

We can rephrase this in terms of rigged partitions. Define the degree of a pair of rigged partitions $(\mu,r;\nu,s)$ to be

$$d(\mu,r;\nu,s) = \deg G_{\mu,\nu} + \sum_{\alpha,i} r_i^{(\alpha)} + \sum_{\alpha,i} s_i^{(\alpha)},$$

and the character of the set $R^{(M,N)}_{m,n}[l_1,l_2,l_3]$ by

$$\mathrm{ch}_q R^{(M,N)}_{m,n}[l_1,l_2,l_3] = \sum_{(\mu,r;\nu,s)\in R^{(M,N)}_{m,n}[l_1,l_2,l_3]} q^{d(\mu,r;\nu,s)}. \tag{3.28}$$

By definition

$$\mathrm{ch}_q \mathcal{G}^{(M,N)}_{m,n}[l_1,l_2,l_3] = \mathrm{ch}_q R^{(M,N)}_{m,n}[l_1,l_2,l_3].$$

Finally, by Theorem 3.5.7, we have

$$\mathrm{ch}_q W^{(M,N)}_k[l_1,l_2,l_3]_{m,n} = \mathrm{ch}_q \mathcal{G}^{(M,N)}_k[l_1,l_2,l_3].$$

Let us compute these characters explicitly. Set

$$A_{\alpha,\beta} = \min(\alpha,\beta).$$

The degree of $\deg G_{\mu,\nu}$ in the space $\mathcal{G}_{\mu,\nu}[l_1,l_2,l_3]$ is given by

$$\begin{aligned} D_{\mu,\nu}[l_1,l_2] &= \sum_\alpha (\alpha-l_1)^+ m_\alpha(\mu) + \sum_\alpha (\alpha-l_2)^+ m_\alpha(\nu) \\ &+ \sum_{\alpha,\beta} A_{\alpha,\beta} m_\alpha(\mu) m_\beta(\mu) + \sum_{\alpha,\beta} A_{\alpha,\beta} m_\alpha(\nu) m_\beta(\nu) - \sum_{\alpha,\beta} A_{\alpha,\beta} m_\alpha(\mu) m_\beta(\nu) \end{aligned} \tag{3.29}$$

In the special case when $l_3 = \min(l_1,l_2)$, we have

$$\tau^{(\alpha,\beta)}[l_1,l_2,\min(l_1,l_2)] \le 0.$$

and thus there is no lower-bound condition on the riggings.

The summation (3.28) with respect to the riggings r,s can be immediately computed using

$$\sum_{0\le r_1\le\ldots\le r_n\le M} q^{r_1+\cdots+r_n} = \begin{bmatrix} M+n \\ n \end{bmatrix},$$

where (if $m\in\mathbb{Z}$ and $n\in\mathbb{Z}_{\ge 0}$) the Gaussian polynomials are

$$\begin{bmatrix} m \\ n \end{bmatrix} = \begin{cases} \frac{\prod_{i=1}^m (1-q^i)}{\prod_{i=1}^n (1-q^i)\prod_{i=1}^{m-n}(1-q^i)} & \text{if } m\ge n; \\ 0 & \text{if } m<n. \end{cases} \tag{3.30}$$

Lemma 3.6.1.

$$\begin{aligned} &\mathrm{ch}_q W^{(M,N)}_k[l_1,l_2,\min(l_1,l_2)]_{m,n} \\ &= \sum_{|\mu|=m,|\nu|=n} q^{D_{\mu,\nu}[l_1,l_2]} \prod_\alpha \begin{bmatrix} P^{(M)}_{\mu,\nu}[l_1]_\alpha + m_\alpha(\mu) \\ m_\alpha(\mu) \end{bmatrix} \prod_\alpha \begin{bmatrix} Q^{(N)}_{\mu,\nu}[l_2]_\alpha + m_\alpha(\nu) \\ m_\alpha(\nu) \end{bmatrix}. \end{aligned}$$

Let $W^{(M,N)}_k[l_1,l_2] = W^{(M,N)}_k[l_1,l_2,\min(l_1,l_2)]$. Then

Theorem 3.6.2. *The character of the space of coinvariants $W^{(M,N)}_k[l_1,l_2]$ is given by*

$$\chi^{(M,N)}_k[l_1,l_2](z_1,z_2,q) = \sum_{\mu,\nu} z_1^{|\mu|} z_2^{|\nu|} q^{D_{\mu,\nu}[l_1,l_2]} \prod_\alpha \begin{bmatrix} P^{(M)}_{\mu,\nu}[l_1]_\alpha + m_\alpha(\mu) \\ m_\alpha(\mu) \end{bmatrix} \prod_\alpha \begin{bmatrix} Q^{(N)}_{\mu,\nu}[l_2]_\alpha + m_\alpha(\nu) \\ m_\alpha(\nu) \end{bmatrix}. \tag{3.31}$$

In what follows, we set $\chi_k^{(M,N)}[l_1,l_2](z_1,z_2,q)=0$ if $l_1<0$ or $l_2<0$.

In Section 6.4 of Part II [FKLMM2], we obtained several identities between the characters of coinvariant spaces for $\widetilde{\mathfrak{H}}$ and $\widehat{\mathfrak{sl}}_2$-modules. One can apply the result above to give "fermionic" formulas for them. For, example, we have

Theorem 3.6.3. *The character (1.1) of the $\widehat{\mathfrak{sl}}_2$ coinvariant space $L_{k,l}^{(M,N)}$ is*

$$\chi_{k,l}^{(M,N)}=z^{-l}\Bigl(\chi_k^{(M+1,N)}[l,k-l](q^{-2}z^2,z^{-2},q)-q\chi_k^{(M+1,N)}[l-1,k-l-1](q^{-2}z^2,z^{-2},q)\Bigr).$$

4. The upper and lower subsets of rigged configurations

In the rest of the paper we prove Theorem 2.2.1. In this section we define admissible pairs (I,J) of subsets of $\{1,\dots,k\}$. Then, we define two kinds of subsets of rigged partitions indexed by admissible pairs, the lower and upper subsets. and construct a bijection from the upper to the lower subsets indexed by the same pair (I,J).

4.1. **Admissibility of** (I,J)**.** For a k-vector $\rho\in\mathbb{Z}^k$, the α-th coordinate of ρ is denoted by ρ_α. The positive and negative parts of ρ, $\rho^\pm\in\mathbb{Z}^k_{\geq 0}$, are defined by $(\rho^\pm)_\alpha=(\rho_\alpha)^\pm$. We have $\rho=\rho^+-\rho^-$.

For k-vectors $\xi,\eta\in\mathbb{Z}^k$ we write $\xi\geq\eta$ if and only if $\xi_\alpha\geq\eta_\alpha$ for all $1\leq\alpha\leq k$. In particular, $\xi\geq 0$ means $\xi_\alpha\geq 0$ for all $1\leq\alpha\leq k$.

For $I\subset\{1,\dots,k\}$, define the k-vectors $\kappa(I)\in\{0,1,\dots,k\}^k$, $\varepsilon(I)\in\{-1,0,1\}^k$ by the formula

$$\kappa(I)_\alpha=\sum_{i\in I,\ i\leq\alpha}1,$$
$$\varepsilon(I)_\alpha=\sum_{i\in I}(\delta_{i,\alpha}-\delta_{i,\alpha+1}),$$

where $\alpha=1,\dots,k$.

We define a partial ordering in the set $2^{\{1,\dots,k\}}$: $J\geq J'$ if and only if $\kappa(J)\geq\kappa(J')$. If we set $J=\{v_1,\dots,v_s\}$ and $J'=\{v'_1,\dots,v'_{s'}\}$ where $v_1<\dots<v_s$ and $v'_1<\dots<v'_{s'}$, this is equivalent to $s\geq s'$ and $v_i\leq v'_i$ for $1\leq i\leq s$.

Sometimes it is convenient to extend the definition of $\kappa(I)$ to I not necessarily satisfying $I\subset\{1,\dots,k\}$. Namely, we use the same definition for $I\subset\{1,2,3,\dots\}$. Note, however, that $\kappa(I)_\alpha=\kappa(I\cap\{1,\dots,k\})_\alpha$ because we consider α only in the region $\{1,\dots,k\}$.

Note that if $I=I_1\bigsqcup I_2$ then $\kappa(I)=\kappa(I_1)+\kappa(I_2)$ and $\varepsilon(I)=\varepsilon(I_1)+\varepsilon(I_2)$. We have $\kappa(I)=\sum_{i\in I}\kappa(i)$ and $\varepsilon(I)=\sum_{i\in I}\varepsilon(i)$, where we denoted $\kappa(i)=\kappa(\{i\})$ and $\varepsilon(i)=\varepsilon(\{i\})$. For example, if $k=5$, we have $\kappa(\{2,4,5\})=(0,1,1,2,3)$ and $\varepsilon(\{2,4,5\})=(-1,1-1,0,1)$. For $\alpha,\beta\in\{1,2,3,\dots\}$, we denote the interval $\{\alpha,\alpha+1,\dots,\beta\}$ by $[\alpha,\beta]$ and the k-vector $\kappa([\alpha,\beta])$ by $\kappa[\alpha,\beta]$.

Fix $0\leq l_1,l_2\leq k$. Let $I=\{u_1,\dots,u_a\}$ $(u_1<\dots<u_a)$ and $J=\{v_1,\dots,v_b\}$ $(v_1<\dots<v_b)$ be subsets of $\{1,\dots,k\}$. We define the (l_1,l_2)-admissibility of (I,J) as follows.

Let $p=p(l_1,J)$ be the number of elements of J which are less than l_1+1,

$$v_1<\dots<v_p<l_1+1\leq v_{p+1}<\dots<v_b.$$

Set

$$t=\max(1,l_1+b-k+1). \tag{4.1}$$

We have $t\leq a$ if and only if $l_1+c<k$.

Lemma 4.1.1. *We can label the complement of $[l_1+1,k]\cap J$ in $[l_1+1,k]$ as follows.*

$$\begin{aligned}&[l_1+1,k]\backslash\{v_{p+1},\dots,v_b\}\\ &=\begin{cases}\{v'_p,\dots,v'_t\} \quad \textit{where } v'_p<\dots<v'_t & \textit{if } l_1+b\geq k;\\ \{v'_p,\dots,v'_1,w_1,\dots,w_{k-l_1-b}\} \quad \textit{where } v'_p<\dots<v'_1<w_1<\dots<w_{k-l_1-b} & \textit{if } l_1+b<k.\end{cases}\end{aligned} \tag{4.2}$$

Proof. Note that $\#\left([l_1+1,k]\backslash\{v_{p+1},\dots,v_b\}\right)=k-l_1-b+p$. If $l_1+b\geq k$ we have $k-l_1-b+p\leq p$, and we label the complement as $v'_p<\dots<v'_t$. If $l_1+b<k$ we have $k-l_1-b+p>p$, and we label the complement as $v'_p<\dots<v'_1<w_1<\dots<w_{k-l_1-b}$. □

In the case $l_1+b\geq k$ it is convenient to set

$$v'_{t-1}=\dots=v'_1=k+1. \tag{4.3}$$

We have

Lemma 4.1.2.

$$\Big(\kappa(J)-\kappa[l_1+1,l_1+b)]\Big)^+=\sum_{i=1}^{p}\left(\kappa(v_i)-\kappa(v'_i)\right). \tag{4.4}$$

Proof. Observe $\kappa_\alpha=(\kappa(J)-\kappa[l_1+1,l_1+b])_\alpha$ when α varies from 1 to k. For $\alpha\leq l_1$, κ increases from $\kappa_{\alpha-1}$ to κ_α by 1 if $\alpha=v_i$ $(1\leq i\leq p)$ or stays constant otherwise. In the case $l_1+b\geq k$, for $l_1+1\leq\alpha\leq k$, κ decreases from $\kappa_{\alpha-1}$ to κ_α by 1 if $\alpha=v'_i$ $(t\leq i\leq p)$ or stays constant otherwise. In the case $l_1+b<k$, for $l_1+1\leq\alpha\leq v'_1$, κ_α decreases by 1 at $\alpha=v'_i$ $(1\leq i\leq p)$ or stays constant otherwise. In particular, we have $\kappa_{v'_1}=0$. For $\alpha>v'_1$, $\kappa_\alpha\leq 0$. The equality (4.4) follows from these observations with the convention (4.3) for $l_1+b\geq k$. □

A pair of subsets (I,J) is called (l_1,l_2)-admissible if

$$a\leq p(l_1,J),\quad b\leq l_2\quad\text{and}\quad v_i\leq u_i<v'_i\ (1\leq i\leq a). \tag{4.5}$$

Note that l_2 appears only in the restriction $b\leq l_2$. A pair $(\emptyset,J)$ is (l_1,l_2)-admissible if and only if $\#(J)\leq l_2$. An (l_1,k)-admissible pair is simply called l_1-admissible. If (I,J) is l_1-admissible, then $\#(I)\leq l_1$.

Note that if $l_1+c\geq k$, (I,J) is l_1 admissible if and only if

$$v_i\leq u_i\quad(1\leq i\leq a).$$

The condition $a\leq p(l_1,J)$ is satisfied because $v_a\leq v_b-c\leq k-c\leq l_1$.

If $l_1+c<k$, for an l_1-admissible pair (I,J) we set

$$\tilde I=\tilde I(I,J)=I\sqcup I' \tag{4.6}$$

$$I'=\begin{cases}\{v'_a,\dots,v'_t\} & \text{if } l_1+b\geq k;\\ \{v'_a,\dots,v'_1\}\sqcup\{w_1,\dots,w_{k-l_1-b}\} & \text{if } l_1+b<k.\end{cases} \tag{4.7}$$

Note that

$$\#(\tilde I)=k-l'_1 \tag{4.8}$$

We also set

$$\tilde J=J\cap[1,v'_a-1]. \tag{4.9}$$

We have

$$\#(\tilde{J}) = v'_a + c - l'_1 - 1 \tag{4.10}$$

because

$$\begin{aligned}
\#(\tilde{J}) &= \#(J \cap [1, l_1]) + \#(J \cap [l_1 + 1, v'_a - 1]) \\
&= p + \#([l_1 + 1, v'_a - 1]) - \#(\{v'_p, \dots, v'_{a+1}\}) \\
&= a + v'_a - l_1 - 1 \\
&= v'_a + c - l'_1 - 1.
\end{aligned}$$

If $l_1 + c \geq k$ we have $v'_a = k + 1$ by (4.3). We set $\tilde{I} = I$ and $\tilde{J} = J$. The equalities (4.8) and (4.10) are valid in this case, too.

Lemma 4.1.3. *Suppose that $l_1 + c < k$. The map $\mathfrak{b} : (I, J) \mapsto (\tilde{I}, \tilde{J})$ given by (4.6) and (4.9) is a bijection between the set of l_1-admissible pairs (I, J) satisfying $\#(I) = a$ and $\#(J) = a + c$ and the set of $(\tilde{I}, \tilde{J})$ satisfying*

$$\tilde{I} = \{u_1, \dots, u_{\tilde{a}}\} \ (\tilde{a} = a + k - l_1 - c), \quad u_1 < \cdots < u_{\tilde{a}}, \quad u_{a+1} \geq l_1 + 1, \tag{4.11}$$

$$\begin{aligned}
&\tilde{J} = \{v_1, \dots, v_{\tilde{b}}\} \ (\tilde{b} = a + u_{a+1} - l_1 - 1), \quad v_1 < \cdots < v_{\tilde{b}}, \\
&v_a \leq l_1, \quad v_{\tilde{b}} < u_{a+1}, \quad v_i \leq u_i \quad (1 \leq i \leq a).
\end{aligned} \tag{4.12}$$

(*In* (4.12), *the condition $v_a \leq l_1$ follows from the others.*)

Proof. We will prove that the inverse map $\mathfrak{c} : (\tilde{I}, \tilde{J}) \mapsto (I, J)$ is given by

$$I = \{u_1, \dots, u_a\}, \quad J = \tilde{J} \sqcup ([u_{a+1}, k] \backslash \tilde{I}). \tag{4.13}$$

Let us prove that the composition $\mathfrak{c} \circ \mathfrak{b}$ is the identity map. Consider (I, J) and $(\tilde{I}, \tilde{J}) = \mathfrak{b}(I, J)$. Set $(I_1, J_1) = \mathfrak{c}(\tilde{I}, \tilde{J})$. Since $u_a < v'_a$, the smallest a elements in $\tilde{I}$ are $u_1 < \cdots < u_a$. Therefore, $I = I_1$.

Note that

$$\begin{aligned}
v'_a &= u_{a+1}, \\
\tilde{J} &= J \cap [1, u_{a+1} - 1], \\
[u_{a+1}, k] \backslash \tilde{I} &= [u_{a+1}, k] \backslash I' \\
&= [u_{a+1}, k] \backslash \Big([u_{a+1}, k] \cap ([l_1 + 1, k] \backslash J) \Big) \\
&= [u_{a+1}, k] \backslash ([u_{a+1}, k] \backslash J) \\
&= [u_{a+1}, k] \cap J.
\end{aligned}$$

Therefore, we have

$$J_1 = \tilde{J} \sqcup ([u_{a+1}, k] \backslash \tilde{I}) = (J \cap [1, u_{a+1} - 1]) \sqcup ([u_{a+1}, k] \cap J) = J.$$

Let us prove that the pair (I, J) given by (4.13) is l_1-admissible. We define u_i and v_j as before from I and J. It is clear that $v_i \leq u_i$ $(1 \leq i \leq a)$.

The number $p = p(l_1, J)$ satisfies $v_{p+1} > l_1$. Since $v_a \leq l_1$, we have $a \leq p$.

Let the smallest $p - a + 1$ elements of the set $[l_1 + 1, k] \backslash J$ be $\{v'_p, \dots, v'_a\}$ $(v'_p < \cdots < v'_a)$. We will show that $v'_a = u_{a+1}$. Then, it follows that $u_i < v'_i$ $(1 \leq i \leq a)$.

Since $v_{\tilde{b}} < u_{a+1}$, we have $\tilde{J} \cap [u_{a+1}, k] = \emptyset$. Then, we have

$$\begin{aligned} [l_1+1, k] \backslash J &= ([l_1+1, u_{a+1}-1] \backslash \tilde{J}) \sqcup \Big([u_{a+1}, k] \backslash ([u_{a+1}, k] \backslash \tilde{I}) \Big) \\ &= ([l_1+1, u_{a+1}-1] \backslash \tilde{J}) \sqcup ([u_{a+1}, k] \cap \tilde{I}). \end{aligned} \tag{4.14}$$

Since $\#([l_1+1, u_{a+1}-1] \backslash \tilde{J}) = u_{a+1} - 1 - l_1 - (\tilde{b} - p) = p - a$, we have $v'_a = u_{a+1}$. □

Fix a, c and I with $\#(I) = a$. In Section 6 we will use the minimal element $J_{\min}$ among J such that $\#(J) = a + c$, (I, J) is l_1-admissible and $\tilde{I}(I, J)$ is fixed.

Lemma 4.1.4. *Suppose that $l_1 + c \geq k$ and fix $I = \{u_1, \ldots, u_a\}$. Consider the set of J such that (I, J) is l_1-admissible and $\#(J) = a + c$. This set has the minimal element given by*

$$J_{\min} = \{\min(u_i, k - b + i)\}_{1 \leq i \leq a} \sqcup [k - c + 1, k]. \tag{4.15}$$

Proof. Set $J = \{v_1, \ldots, v_b\}$ $(v_1 < \cdots < v_b)$. We have obviously $v_i \leq k - b + i$ $(1 \leq i \leq b)$. For $1 \leq i \leq a$ we have further $v_i \leq u_i$. Therefore, the minimal element is given by (4.15). □

For $l_1 + c < k$, we obtain $J_{\min}$ by using the bijection $\mathfrak{b}$.

Lemma 4.1.5. *Suppose that $l_1 + c < k$ and fix $\tilde{I}$ satisfying (4.11). Set $I = \{u_1, \ldots, u_a\}$ and consider the set of J such that (I, J) is l_1-admissible, $\#(J) = a + c$ and $\tilde{I}(I, J) = \tilde{I}$. This set has the minimal element given by*

$$J_{\min} = \{\min(u_i, l_1 - a + i)\}_{1 \leq i \leq a} \sqcup [l_1 + 1, u_{a+1} - 1] \sqcup ([u_{a+1}, k] \backslash \tilde{I}). \tag{4.16}$$

Proof. The minimal set $\tilde{J}_{\min}$ among $\tilde{J}$ satisfying (4.12) is given by $\tilde{J}_{\min} = \{\min(u_i, l_1 - a + i)\}_{1 \leq i \leq a} \sqcup [l_1 + 1, u_{a+1} - 1]$. Then, $J_{\min}$ is given by (4.13). □

4.2. **Vectors ρ and σ and lower subsets.** For an (l_1, l_2)-admissible pair (I, J), define the vectors $\rho(I, J) = \rho_{l_1,k}(I, J)$, $\sigma(J) = \sigma_{l_2,k}(J) \in \mathbb{Z}^k$ by

$$\rho(I, J) = \sum_{i=1}^{a} (\kappa(v_i) - \kappa(u_i)) + \sum_{i=a+1}^{p} (\kappa(v_i) - \kappa(v'_i)), \tag{4.17}$$

$$\sigma(J) = \kappa[1, l_2] - \kappa(J). \tag{4.18}$$

Note that $\rho(I, J), \sigma(J) \geq 0$.

We introduce a few notations.

We use the symbols

$$\leq_\varepsilon \quad \leftrightarrow \quad \begin{cases} \leq & \text{if } \varepsilon \neq 1; \\ = & \text{if } \varepsilon = 1, \end{cases} \tag{4.19}$$

$$\leq^\varepsilon \quad \leftrightarrow \quad \begin{cases} \leq & \text{if } \varepsilon \neq -1; \\ = & \text{if } \varepsilon = -1. \end{cases} \tag{4.20}$$

For a rigging $r = \{r_i^{(\alpha)}\}_{\substack{1 \leq \alpha \leq k \\ 1 \leq i \leq m_\alpha}}$, we define

$$r[\alpha] = \begin{cases} r^{(\alpha)}_{m_\alpha} & \text{if } m_\alpha \geq 1; \\ \infty & \text{if } m_\alpha = 0. \end{cases} \tag{4.21}$$

For a pair of subsets (I,J) $(I,J\subset\{1,\dots,k\})$ and a pair of integers (l_1,l_2) $(0\le l_1,l_2\le k)$, we define the subset $R_{m,n}[l_1,l_2]_{I,J}\subset R_{m,n}$ as follows. If (I,J) is (l_1,l_2)-admissible, we set

$$\begin{aligned} R_{m,n}[l_1,l_2]_{I,J}=\{&(\mu,r;\nu,s)\in R_{m,n};\\ &r[\alpha]\ge_{\varepsilon(I)_\alpha}\rho(I,J)_\alpha\quad\text{and}\quad s[\alpha]\ge_{\varepsilon(J)_\alpha}\sigma(J)_\alpha\quad\text{for all }1\le\alpha\le k\}\end{aligned}\tag{4.22}$$

(see (4.19)). Otherwise we set $R_{m,n}[l_1,l_2]_{I,J}=\emptyset$.

The restriction for $r[\alpha]$ is called marked if $\varepsilon(I)_\alpha=1$ and, therefore, it takes the form $\rho(I,J)_\alpha=r[\alpha]$; it is called unmarked otherwise, namely, if it takes the form $\rho(I,J)_\alpha\le r[\alpha]$. Similarly, we distinguish the marked and unmarked restrictions for $s[\alpha]$.

Suppose that $(\mu,r;\nu,s)$ is contained in $R_{m,n}[l_1,l_2]_{I,J}$. Then, $m_\alpha\ne0$ if $\varepsilon(I)_\alpha=1$; $n_\alpha\ne0$ if $\varepsilon(J)_\alpha=1$.

For $M,N\ge0$ we define

$$R^{(M,N)}_{m,n}[l_1,l_2]_{I,J}=R_{m,n}[l_1,l_2]_{I,J}\cap R^{(M,N)}_{m,n}[l_1,l_2].\tag{4.23}$$

If an element $(\mu,r;\nu,s)$ is contained in $R^{(M,N)}_{m,n}[l_1,l_2]_{I,J}$, and if $m_\alpha\ne0$ for some α, then we have $\rho(I,J)_\alpha\le P^{(M)}_{\mu,\nu}[l_1]_\alpha$; if $n_\alpha\ne0$ then $\sigma(J)_\alpha\le Q^{(N)}_{\mu,\nu}[l_2]_\alpha$. In the rest of this section we prove the validity of these inequalities when $m_\alpha=0$ or $n_\alpha=0$.

Let us abbreviate $P^{(M)}_{\mu,\nu}[l_1]_\alpha$ to P_α, and $\rho(I,J)_\alpha$ to ρ_α. Recall that

$$\rho(I,J)=\sum_{\alpha=1}^{p}\kappa(v_\alpha)-\sum_{\alpha=1}^{a}\kappa(u_\alpha)-\sum_{\alpha=a+1}^{p}\kappa(v'_\alpha),\tag{4.24}$$

where $v_1<\dots<v_p\le l_1$, $v'_p<\dots<v'_t<v'_{t-1}=\dots=v'_1=k+1$, $u_1<\dots<u_a$ and $v_i\le u_i<v'_i$ $(1\le i\le a)$. Here $t=\max(1,l_1+b-k+1)$.

We set

$$J_{\rm up}=\{v_1,\dots,v_p\}=J\cap[1,l_1],\tag{4.25}$$

$$J'_{\rm down}=\begin{cases}\{v'_p,\dots,v'_t\} & \text{if } l_1+c\ge k;\\ \{v'_p,\dots,v'_{a+1}\} & \text{if } l_1+c<k.\end{cases}\tag{4.26}$$

If $l_1+c\ge k$ we have $t>a$ and

$$J'_{\rm down}=[l_1+1,k]\backslash J.\tag{4.27}$$

If $l_1+c<k$ we have $t\le a$ and

$$J'_{\rm down}\cup\Big(\{\alpha;\alpha\ge l_1+1,\rho_\alpha=0\}\backslash J\Big)=[l_1+1,k]\backslash J.\tag{4.28}$$

We list a few more properties of ρ_α.

$(P1)$ $\rho_\alpha-2\le\rho_{\alpha+1}\le\rho_\alpha+1$,

$(P2)$ If $\alpha+1\in J_{\rm up}$, then $\rho_{\alpha+1}\ge\rho_\alpha$,

$(P3)$ If $\alpha+1\notin J_{\rm up}$, then $\rho_{\alpha+1}\le\rho_\alpha$,

$(P4)$ If $\alpha+1\in J'_{\rm down}$, then $\rho_{\alpha+1}\le\rho_\alpha-1$,

$(P5)$ If $\alpha+1\notin J'_{\rm down}$, then $\rho_{\alpha+1}\ge\rho_\alpha-1$.

Lemma 4.2.1. *If $R^{(M,N)}_{m,n}[l_1,l_2]_{I,J}$ contains an element $(\mu,r;\nu,s)$, then we have $\rho_k\le P_k$..*

Proof. Assume that $P_i \geq \rho_i$ and $P_\alpha < \rho_\alpha$ $(i+1 \leq \alpha \leq k)$ for some $0 \leq i \leq k-1$. As we noted at the beginning of this section we have $m_\alpha = 0$ $(i+1 \leq \alpha \leq k)$. This implies $\varepsilon(I)_\alpha \neq 1$ $(i+1 \leq \alpha \leq k)$. Therefore, we have

$$I \subset [1, i]. \tag{4.29}$$

We have

$$\begin{aligned} P_{i+1} - P_i &= M - (i+1-l_1)^+ + (i-l_1)^+ + \sum_{\beta \geq i+1} n_\beta \\ &< \rho_{i+1} - \rho_i. \end{aligned} \tag{4.30}$$

Subcase 1 : $i+1 \leq l_1$.

From $(P1)$ we have $\rho_{i+1} - \rho_i \leq 1$. Using (4.30) we have

$$P_{i+1} - P_i = M + \sum_{\beta \geq i+1} n_\beta < 1.$$

Therefore we have $M = 0$ and $n_{i+1} = \cdots = n_k = 0$. This implies $J \subset [1, i]$, and therefore $i+1 \notin J$. Using $(P3)$ we have $\rho_{i+1} \leq \rho_i$. This is a contradiction because

$$0 = P_{i+1} - P_i < \rho_{i+1} - \rho_i \leq 0.$$

Subcase 2 : $i+1 \geq l_1 + 1$.

We have $i+1 \notin J_{\rm up}$ because $J_{\rm up} \subset [1, l_1]$. From $(P3)$ follows $\rho_{i+1} - \rho_i \leq 0$ and using (4.30) we have

$$P_{i+1} - P_i = M - 1 + \sum_{\beta \geq i+1} n_\beta < 0.$$

Therefore, we have $M = 0$, $n_{i+1} = \cdots = n_k = 0$ and $i+1 \notin J$ again.

If $l_1 + c \geq k$, because of (4.27) we have $i+1 \in J'_{\rm down}$. Using $(P4)$ we have $\rho_{i+1} - \rho_i \leq -1$. This is a contradiction because

$$-1 = P_{i+1} - P_i < \rho_{i+1} - \rho_i \leq -1.$$

If $l_1 + c < k$, we proceed as follows. If $i+1 \in J'_{\rm down}$, it leads to a contradiction as above. If $i+1 \notin J'_{\rm down}$, because of (4.28) we have $\rho_{i+1} = 0$. It implies $P_{i+1} < 0$. However, this is prohibited by (2.8). □

Lemma 4.2.2. *If $R^{(M,N)}_{m,n}[l_1, l_2]_{I,J}$ contains an element $(\mu, r; \nu, s)$, then we have $\rho_\alpha \leq P_\alpha$ $(1 \leq \alpha < k)$.*

Proof. We lead to a contradiction assuming that for some i and j satisfying $1 \leq i+1 < j \leq k$ we have

$$P_i \geq \rho_i, \quad P_\alpha < \rho_\alpha \ (i+1 \leq \alpha \leq j-1), \quad P_j \geq \rho_j.$$

We set $p = \frac{1}{j-i}$ so that $pi + (1-p)j = j-1$. A simple calculation as (2.13) shows

$$P_{j-1} - \rho_{j-1} \geq \Delta\rho + \theta(i < l_1 < j)(l_1 - i)p + \sum_{i<\beta<j} (\beta - i) p n_\beta, \tag{4.31}$$

$$\Delta\rho = p\rho_i + (1-p)\rho_j - \rho_{j-1}. \tag{4.32}$$

Here we used $m_\beta = 0$ for $i < \beta < j$. Note that the last two terms in the RHS of (4.31) is non-negative.

We consider three cases $\rho_j \geq \rho_{j-1}$, $\rho_j = \rho_{j-1} - 1$ and $\rho_j = \rho_{j-1} - 2$, separately.
Case 1 : $\rho_j \geq \rho_{j-1}$.

Because of $(P1)$ we have $\rho_{j-1} \leq \rho_i + j - i - 1$. From this follows

$$\Delta\rho \geq p(\rho_{j-1} - (j - i - 1)) + (1 - p)\rho_{j-1} - \rho_{j-1} = -1 + p.$$

Using (4.31) we have $P_{j-1} - \rho_{j-1} \geq 0$, which is a contradiction.
Case 2 : $\rho_j = \rho_{j-1} - 1$.
Subcase 1 : $i \geq l_1$.

Using $(P3)$ and (4.25) we have $\rho_{j-1} \leq \rho_i$. Then, we have

$$P_{j-1} - \rho_{j-1} \geq \Delta\rho \geq p\rho_{j-1} + (1 - p)(\rho_{j-1} - 1) - \rho_{j-1} = -1 + p. \tag{4.33}$$

This is a contradiction.
Subcase 2 : $i < l_1 < j$.

Because of (P3) and (4.25) we have $\rho_{j-1} \leq \rho_i + l_1 - i$. Therefore, noting that $\theta(i < l_1 < j) = 1$, we have again

$$\begin{aligned} P_{j-1} - \rho_{j-1} &\geq p(\rho_{j-1} - (l_1 - i)) + (1 - p)(\rho_{j-1} - 1) - \rho_{j-1} + (l_1 - i)p \\ &= -1 + p. \end{aligned}$$

This is a contradiction.
Subcase 3 : $j \leq l_1$.

We have $j \notin J$ because otherwise $j \in J_{\mathrm{up}}$ and and using $(P2)$ we have $\rho_j \geq \rho_{j-1}$, which is a contradiction.

We will prove by induction the following statements for $i + 1 \leq \alpha \leq j - 1$:

$$\begin{aligned} &(C1)_\alpha \qquad n_\alpha = 0, \\ &(C2)_\alpha \qquad [\alpha, j] \cap J = \emptyset, \\ &(C3)_\alpha \qquad \rho_i \geq \rho_{j-1} - (\alpha - i - 1). \end{aligned}$$

Then, $(C3)_{i+1}$ leads to (4.33), which is a contradiction.

We first note that $(C2)_j$ and $(C3)_j$ are valid. These are the basis for the induction. From $(C1)_\alpha$ follows $\varepsilon(J)_\alpha \neq 1$. Using $(C2)_{\alpha+1}$ we have $\alpha \notin J$, and therefore $(C2)_\alpha$. Because of $(P3)$ from $(C2)_\alpha$ follows $(C3)_\alpha$. Finally, we show that for $i + 2 \leq \alpha \leq j$ from $(C3)_\alpha$ follows $(C1)_{\alpha-1}$. Unless $n_{\alpha-1} = 0$ we have again

$$\begin{aligned} P_{j-1} - \rho_{j-1} &\geq p(\rho_{j-1} - (\alpha - i - 1)) + (1 - p)(\rho_{j-1} - 1) - \rho_{j-1} + (\alpha - 1 - i)pn_{\alpha-1} \\ &\geq -1 + p. \end{aligned}$$

Case 3 : $\rho_j = \rho_{j-1} - 2$.
Subcase 1 : $i \geq l_1$.

We will prove by induction the following statements for $i + 1 \leq \alpha \leq j - 1$.

$$\begin{aligned} &(C1)'_\alpha \qquad n_\alpha = 0, \\ &(C2)'_\alpha \qquad [\alpha, j] \subset J'_{\mathrm{down}} \\ &(C3)'_\alpha \qquad \rho_i \geq \rho_{j-1} + j - \alpha. \end{aligned}$$

Then, from $(C3)'_{i+1}$, we have $\rho_i \geq \rho_{j-1} + j - i - 1$. Using this we have

$$\begin{aligned} P_{j-1} - \rho_{j-1} &\geq p(\rho_{j-1} + j - i - 1) + (1 - p)(\rho_{j-1} - 2) - \rho_{j-1} \\ &= -1 + p. \end{aligned}$$

This is a contradiction.

As we have noted above we have $(C2)'_j$. Because of $(P3)$ and (4.25), we have $\rho_i \geq \rho_{j-1}$. This is $(C3)'_j$.

Assume that $(C1)'_\alpha$ and $(C2)'_{\alpha+1}$ are valid for some $i+1 \leq \alpha \leq j-1$. From $(C1)'_\alpha$ follows $\varepsilon(J)_\alpha \neq 1$. Since $\alpha+1 \notin J$ by $(C2)'_{\alpha+1}$, we have $\alpha \notin J$.

If $l_1 + c \geq k$, because of (4.27) we have $\alpha \in J'_{\text{down}}$. If $l_1 + c < k$, we use (4.28). Note that $\alpha \geq i+1 \geq l_1+1$ and $\alpha \notin J$. If $\rho_\alpha = 0$, we have $P_\alpha < 0$, which contradicts (2.8). Otherwise, we have $\alpha \in J'_{\text{down}}$. Thus we have derived $(C2)'_\alpha$ from $(C1)'_\alpha$ and $(C2)'_{\alpha+1}$.

Using (P3) and (P4) we can derive $(C3)'_\alpha$ from $(C2)'_\alpha$.

Suppose that we have $(C3)'_\alpha$ for some $i+2 \leq \alpha \leq j$. Unless $n_{\alpha-1} = 0$ we have

$$\begin{aligned} P_{j-1} - \rho_{j-1} &\geq p(\rho_{j-1} + j - \alpha) + (1-p)(\rho_{j-1} - 2) - \rho_{j-1} + (\alpha - i - 1)pn_{\alpha-1} \\ &\geq -1 + p. \end{aligned}$$

This is a contradiction. We have derived $(C1)'_{\alpha-1}$ from $(C3)'_\alpha$.

Subcase 2 : $i < l_1 < j$.

We will prove by induction the following statements for $i+1 \leq \alpha \leq j-1$.

$$\begin{array}{ll} (C1)''_\alpha & n_\alpha = 0, \\ (C2)''_\alpha & [\alpha, j] \cap J = \emptyset, \\ (C3)''_\alpha & \rho_i \geq \rho_{j-1} - l_1 - \alpha + i + j. \end{array}$$

Then, from $(C3)''_{i+1}$ we have $\rho_i \geq \rho_{j-1} + j - l_1 - 1$. Therefore we have

$$P_{j-1} - \rho_{j-1} \geq p(\rho_{j-1} + j - l_1 - 1) + (1-p)(\rho_{j-1} - 2) - \rho_{j-1} + (l_1 - i)p = -1 + p,$$

which is a contradiction.

We have $(C2)''_j$ and $(C3)''_j$. It is obvious that from $(C1)''_\alpha$ and $(C2)''_{\alpha+1}$ follows $(C2)''_\alpha$.

Suppose that $(C2)''_\alpha$ is valid for some $i+1 \leq \alpha \leq j-1$. In particular, we have $\alpha \notin J$. If $\alpha \geq l_1 + 1$, using (4.27) or (4.28) we have $\alpha \in J'_{\text{down}}$ unless we have $l_1 + c < k$ and $P_\alpha < \rho_\alpha = 0$, which contradicts (2.8). Therefore, by using (P4) (if $\alpha \geq l_1 + 1$) or (P3) (if $\alpha \leq l_1$) we have $(C3)''_\alpha$.

Suppose that $(C3)''_\alpha$ is valid for some $i+2 \leq \alpha \leq j$. Unless $n_{\alpha-1} = 0$ from (4.31) we have

$$\begin{aligned} P_{j-1} - \rho_{j-1} &\geq p(\rho_{j-1} - l_1 - \alpha + i + j)) + (1-p)(\rho_{j-1} - 2) - \rho_{j-1} \\ &\quad + (l_1 - i)p + (\alpha - i - 1)pn_{\alpha-1} \\ &\geq -1 + p. \end{aligned}$$

This is a contradiction. Thus, we have proved $(C1)''_{\alpha-1}$.

Subcase 3 : $l_1 \geq j$.

Because of $(P5)$ we have $j \in J'_{\text{down}}$. Because of (4.27), this is a contradiction. $\square$

Next we proceed to the inequality $\sigma(J) \leq Q^{(N)}_{\mu,\nu}[l_2]$. Let us abbreviate $Q^{(N)}_{\mu,\nu}[l_2]_\alpha$ to Q_α and $\sigma(J)_\alpha$ to σ_α. Recall that $b \leq l_2$ and

$$\sigma(J) = \sum_{\alpha=1}^{l_2} \kappa(\alpha) - \sum_{i=1}^{b} \kappa(v_i). \tag{4.34}$$

We have, in particular, $\sigma_\alpha - 1 \leq \sigma_{\alpha+1} \leq \sigma_\alpha + 1$.

Lemma 4.2.3. *Suppose that $N \geq 1$. If $R^{(M,N)}_{m,n}[l_1, l_2]_{I,J}$ contains an element $(\mu, r; \nu, s)$, then we have $\sigma_k \leq Q_k$.*

Proof. Assume that $Q_i \geq \sigma_i$ for some $0 \leq i \leq k-1$ and $Q_\alpha < \sigma_\alpha$ $(i+1 \leq \alpha \leq k)$. We have $n_\alpha = 0$ $(i+1 \leq \alpha \leq k)$. This implies $\varepsilon(J)_\alpha \neq 1$ $(i+1 \leq \alpha \leq k)$. Therefore, we have $J \subset [1, i]$.

We have

$$\begin{aligned} Q_{i+1} - Q_i &= N - (i+1-l_2)^+ + (i-l_2)^+ + \sum_{\beta \geq i+1} m_\beta \\ &< \sigma_{i+1} - \sigma_i. \end{aligned} \tag{4.35}$$

Subcase 1 : $i + 1 \leq l_2$.

From (4.34) we have $\sigma_{i+1} - \sigma_i \leq 1$. Using (4.35) we have

$$Q_{i+1} - Q_i = N + \sum_{\beta \geq i+1} m_\beta < 1.$$

This is a contradiction because we assumed $N - 1 \geq 0$.

Subcase 2 : $l_2 \leq i$.

We have

$$Q_{i+1} - Q_i = N - 1 + \sum_{\beta \geq i+1} m_\beta < 0.$$

This is a contradiction. □

Lemma 4.2.4. *Suppose that $N \geq 1$. If $R^{(M,N)}_{m,n}[l_1, l_2]_{I,J}$ contains an element $(\mu, r; \nu, s)$, then we have $\sigma_\alpha \leq Q_\alpha$ $(1 \leq \alpha < k)$.*

Proof. Suppose that for some i and j such that $1 \leq i+1 < j \leq k$ we have $Q_i \geq \sigma_i$, $Q_\alpha < \sigma_\alpha$ $(i+1 \leq \alpha \leq j-1)$ and $Q_j \geq \sigma_j$. We set $p = \frac{1}{j-i}$. We have

$$Q_{j-1} - \sigma_{j-1} \geq \Delta\sigma + \theta(i < l_2 < j)(l_2 - i)p + \sum_{i<\beta<j} (\beta - i) p m_\beta, \tag{4.36}$$

$$\Delta\sigma = p\sigma_i + (1-p)\sigma_j - \sigma_{j-1}. \tag{4.37}$$

Case 1 : $\sigma_j \geq \sigma_{j-1}$.

We have

$$\Delta\sigma \geq p\Big(\sigma_{j-1} - (j-i-1)\Big) + (1-p)\sigma_{j-1} - \sigma_{j-1} = -1 + p.$$

Using (4.36) we have $Q_{j-1} - \sigma_{j-1} \geq 0$, which is a contradiction.

Case 2 : $\sigma_j = \sigma_{j-1} - 1$.

From (4.34) we have $l_2 + 1 \leq j$ and $\sigma_{j-1} \leq \sigma_j + (l_2 - i)^+$. Therefore, we have

$$Q_{j-1} - \sigma_{j-1} \geq p\Big(\sigma_{j-1} - (l_2 - i)^+\Big) + (1-p)(\sigma_{j-1} - 1) - \sigma_{j-1} + (l_2 - i)^+ p = -1 + p,$$

which is a contradiction. □

We have proved

Proposition 4.2.5. *Suppose that $M, N-1 \geq 0$. If $R^{(M,N)}_{m,n}[l_1, l_2]_{I,J}$ contains an element $(\mu, r; \nu, s)$, then we have*

$$\rho(I, J) \leq P^{(M)}_{\mu,\nu}[l_1], \quad \sigma(J) \leq Q^{(N)}_{\mu,\nu}[l_2]. \tag{4.38}$$

4.3. **Vectors ρ' and σ' and upper subsets.** The basic idea in Theorem 2.2.1 is to change the rigged partitions with degrees $(M, N-1)$ to those with degrees (M, N). The parameters (I, J) describes the change of the partitions from (μ', ν') given by m'_α, n'_α to (μ, ν) given by m_α, n_α:

$$m_\alpha = m'_\alpha + \varepsilon(I)_\alpha, \quad n_\alpha = n'_\alpha + \varepsilon(J)_\alpha \ (1 \le \alpha \le k). \tag{4.39}$$

The corresponding change in the riggings is described by the change of the upper bounds:

$$\Delta r = P^{(M)}_{\mu,\nu}[l_1] - P^{(M)}_{\mu',\nu'}[l'_1] = \kappa(J) - 2\kappa(I) + \kappa[l'_1+1, k] - \kappa[l_1+1, k], \tag{4.40}$$

$$\Delta s = Q^{(N)}_{\mu,\nu}[l_2] - Q^{(N-1)}_{\mu',\nu'}[l'_2] = \kappa(I) - 2\kappa(J) + \kappa[1, l_2] + \kappa[l'_2+1, k]. \tag{4.41}$$

Here l'_1, l'_2 are given by (2.16). Note that the results are not explicitly dependent on (μ', ν') or (μ, ν). They are determined only by $I, J, l_1, l_2, l'_1, l'_2$.

The vectors ρ and σ give the lower bounds to the riggings in the lower subsets. We define the upper subsets by using the shifted lower bounds ρ' and σ'. Naturally, the shifts are given by Δr and Δs.

For an l_1-admissible pair (I, J) such that $\#(I) = a$ and $\#(J) = b = a + c$, we define the vectors $\rho'(I, J), \sigma'(I, J) \in \mathbb{Z}^k$ by

$$\begin{aligned}
\rho'(I,J) &= \rho(I,J) - \Delta r \\
&= \kappa(I) + \kappa[l_1+1, k] - \sum_{i=p+1}^{b} \kappa(v_i) - \sum_{i=a+1}^{p} \kappa(v'_i) - \kappa[l'_1+1, k] \\
&= \kappa(\tilde{I}) - \kappa[l'_1+1, k], \qquad (4.42)\\
\sigma'(I,J) &= \sigma(J) - \Delta s \\
&= \kappa(J) - \kappa(I) - \kappa[l'_2+1, k], \qquad (4.43)
\end{aligned}$$

where we use $\tilde{I}$ defined in Section 4.1.

The following is clear from (4.43) and (4.42).

Lemma 4.3.1. *We have*

$$\rho'(I,J), \sigma'(I,J) \ge 0 \quad \textit{and} \quad \rho'(I,J)_k = \sigma'(I,J)_k = 0. \tag{4.44}$$

The following lemma will be used in Section 6. We follow the setting in Lemmas 4.1.4 and 4.1.5.

Lemma 4.3.2. *We have*

$$\sigma'(I, J_{\min}) = \begin{cases} \Big(\kappa[k-b+1, k-c] - \kappa(I)\Big)^+ & \textit{if } l_1 + c \ge k; \\ \Big(\kappa[l_1-a+1, k-c] - \kappa(\tilde{I})\Big)^+ & \textit{if } l_1 + c < k. \end{cases} \tag{4.45}$$

Proof. If $l_1 + c \ge k$, using (4.43) and (4.15) we have

$$\begin{aligned}
\sigma'(I, J_{\min}) &= \sum_{i=1}^{a} \kappa(\min(u_i, k-b+i)) - \kappa(I) \\
&= \Big(\kappa[k-b+1, k-c] - \kappa(I)\Big)^+. \qquad (4.46)
\end{aligned}$$

If $l_1 + c < k$, set $\tilde{I} = I \sqcup I'$. We have $I' \subset [u_{a+1}, k]$ and $[u_{a+1}, k] \backslash \tilde{I} = [u_{a+1}, k] \backslash I'$. Therefore, using (4.43) and (4.16) we have

$$\begin{aligned} \sigma'(I, J_{\min}) &= \sum_{i=1}^{a} \kappa(\min(u_i, l_1 - a + i)) - \kappa(\tilde{I}) + \kappa[l_1 + 1, l_2'] \\ &= \Big(\kappa[l_1 - a + 1, l_2'] - \kappa(\tilde{I})\Big)^+ . \end{aligned} \tag{4.47}$$

□

For l_1, a, b ($0 \le a \le b$) and (I, J) such that $\#(I) = a$ and $\#(J) = b$, we define the subset $R_{m-a,n-b}[l_1]^{I,J} \subset R_{m-a,n-b}$ as follows. If (I, J) is l_1-admissible, we set

$$\begin{aligned} R_{m-a,n-b}[l_1]^{I,J} = \{&(\mu', r'; \nu', s') \in R_{m-a,n-b}; \\ &\rho'(I, J)_\alpha \le^{\varepsilon(I)_\alpha} r'[\alpha] \quad \text{and} \quad \sigma'(I, J)_\alpha \le^{\varepsilon(J)_\alpha} s'[\alpha] \quad \text{for all } 1 \le \alpha \le k\} \end{aligned} \tag{4.48}$$

(see (4.20) and (4.21)). Otherwise, we set $R_{m-a,n-b}[l_1]^{I,J} = \emptyset$.

If $l_1 + c \ge k$, the set $R_{m-a,n-b}[l_1]^{I,J}$ is independent of l_1. Sometimes we abbreviate $R_{m-a,n-b}[l_1]^{I,J}$ to $R^{I,J}_{m-a,n-b}$ in this case in avoiding confusion caused by the presence of l_1 in the written formulas.

We call the marking of ρ', σ' as before. The restrictions (4.48) for $r'[\alpha]$ or $s'[\alpha]$ are marked if and only if $\varepsilon(I)_\alpha = -1$ or $\varepsilon(J)_\alpha = -1$, respectively, and hence they are equalities.

For $M, N - 1 \ge 0$ we define

$$R^{(M,N-1)}_{m-a,n-b}[l_1]^{I,J} = R_{m-a,n-b}[l_1]^{I,J} \cap R^{(M,N-1)}_{m-a,n-b}[l_1', l_2']. \tag{4.49}$$

We have

Lemma 4.3.3. *For $I, J \subset \{1, \dots, k\}$ such that $\#(I) = a, \#(J) = b$, if $R^{(M,N-1)}_{m-a,n-b}[l_1]^{I,J}$ contains an element $(\mu', r'; \nu', s')$, then we have*

$$\rho'(I, J)_\alpha \le P^{(M)}_{\mu',\nu'}[l_1']_\alpha, \quad \sigma'(I, J)_\alpha \le Q^{(N-1)}_{\mu',\nu'}[l_2']_\alpha. \tag{4.50}$$

Proof. The proof is completely parallel to Proposition 4.2.5 (we use (4.43) and (4.42)) except that the inequalities (4.50) for $\alpha = k$ follow directly from (2.8) and (4.44). □

4.4. **Bijection.** Define the map

$$\mathfrak{m}_{I,J} : R^{(M,N-1)}_{m-a,n-b}[l_1]^{I,J} \to R^{(M,N)}_{m,n}[l_1, l_2]_{I,J}$$

by the formula

$$\mathfrak{m}_{I,J}(\mu', r'; \nu', s') = (\mu, r, \nu, s),$$

where

$$\mu = \mu' + \varepsilon(I), \quad \nu = \nu' + \varepsilon(J), \tag{4.51}$$

and the riggings r, s are defined by

$$\begin{aligned} r^{(\alpha)}_i &= r'^{(\alpha)}_i + (\Delta r)_\alpha \quad (1 \le i \le m'_\alpha - 1); \\ r^{(\alpha)}_{m'_\alpha} &= r'^{(\alpha)}_{m'_\alpha} + (\Delta r)_\alpha \qquad \text{if } \varepsilon(I)_\alpha = 0, 1; \\ r^{(\alpha)}_{m'_\alpha + 1} &= \rho(I, J)_\alpha \qquad \text{if } \varepsilon(I)_\alpha = 1, \end{aligned}$$

and

$$\begin{aligned} s_i^{(\alpha)} &= {s'}_i^{(\alpha)} + (\Delta s)_\alpha \quad (1 \le i \le n'_\alpha - 1);\\ s_{n'_\alpha}^{(\alpha)} &= {s'}_{n'_\alpha}^{(\alpha)} + (\Delta s)_\alpha \quad \text{if } \varepsilon(J)_\alpha = 0, 1;\\ s_{n'_\alpha+1}^{(\alpha)} &= \sigma(J)_\alpha \qquad \text{if } \varepsilon(J)_\alpha = 1. \end{aligned}$$

We conclude this section by proving

Proposition 4.4.1. *For any $I, J \subset \{i, \dots, k\}$ the map $\mathfrak{m}_{I,J}$ is a bijection.*

Proof. It is enough to show the bijectivity of $\mathfrak{m}_{I,J}$ between the subset of $R^{(M,N-1)}_{m-a,n-b}[l_1]^{I,J}$ with a fixed μ', ν' and the subset of $R^{(M,N)}_{m,n}[l_1, l_2]_{I,J}$ with μ, ν given by (4.51). Because of Lemma 4.2.5 and Lemma 4.3.3, and the definitions (4.40),(4.41),(4.42) and (4.43), these two subsets are both empty or the inequalities (4.38) and (4.50) are both valid. In both cases, the bijectivity is clear. □

5. Decomposition of $R^{(M,N)}_{m,n}[l_1, l_2, l_3]$

Fix k, l_1, l_2 and l_3 as (2.3). The aim of this section is to decompose the set $R^{(M,N)}_{m,n}[l_1, l_2, l_3]$ as

$$R^{(M,N)}_{m,n}[l_1, l_2, l_3] = \bigsqcup_{\substack{I,J\\ \#(I)\le l_3, \#(J)\le l_2}} R^{(M,N)}_{m,n}[l_1, l_2]_{I,J}.$$

Namely, we decompose the left hand side, in which the riggings r and s are restricted from below by the condition (2.5), into the subsets in the right hand side, in which the riggings are restricted from below separately for each $r[\alpha]$ and $s[\alpha]$ according to (I, J).

In fact, it is enough to decompose $R_{m,n}[l_1, l_2, l_3]$ as

$$R_{m,n}[l_1, l_2, l_3] = \bigsqcup_{\substack{I,J\\ \#(I)\le l_3, \#(J)\le l_2}} R_{m,n}[l_1, l_2]_{I,J}.$$

The proof will be carried out in two steps.

The first step is to take the union of the sets $R_{m,n}[l_1, l_2]_{I,J}$ over I for a fixed J. The is done in Lemma 5.1.1; the union is denoted by $R_{m,n}[l_1, l_2, l_3]_J$. The idea of the proof is simple. For a given non-negative integer t the set of integers $\{i; i \ge t\}$ is the disjoint union of $\{i; i \ge t+1\}$ and $\{i; i = t\}$. We need more elaborate arguments in the proof. However, it is done by a successive application of this simple fact.

The second step is to take the union of the sets $R_{m,n}[l_1, l_2, l_3]_J$ over J and obtain $R_{m,n}[l_1, l_2, l_3]$. First we carry out this step for $l_3 = \min(l_1, l_2)$. This is actually a special case of the first step. We obtain $R_{m,n} = R_{m,n}[l_1, l_2, \min(l_1, l_2)]$ as the union. Then, we show that the complement in $R_{m,n}$ of the union of $R_{m,n}[l_1, l_2, l_3]_J$ is equal to the union of its complement in $R_{m,n}[l_1, l_2, \min(l_1, l_2)]_J$. This is done by using another simple fact that the complement $\{i; i \ge 0\}\backslash\{i; i \ge t\}$ is the union of $\{i; i = s\}$ for $0 \le s \le t-1$.

5.1. Union of $R_{m,n}[l_1, l_2]_{I,J}$ over I.

We denote $\#(I) = a$ and $\#(J) = b$ as before. Given J such that $b \le l_2$, we set $p = p(l_1, J)$ as in Section 4.1. Define

$$I_{\max}(J) = \{v_1, \dots, v_{\min(l_3,p)}\}, \quad \rho_{\max}(J) = \rho(I_{\max}(J), J), \tag{5.1}$$

$$\begin{aligned} &R_{m,n}[l_1, l_2, l_3]_J = \{(\mu, r; \nu, s) \in R_{m,n};\\ &r[\alpha] \ge \rho_{\max}(J)_\alpha (1 \le \alpha \le k), s[\alpha] \ge_{\varepsilon(J)_\alpha} \sigma(J)_\alpha (1 \le \alpha \le k)\}. \end{aligned} \tag{5.2}$$

We also define the subset of $2^{\{1,\dots,k\}}$:

$$T^{(k)}(J;l_1,l_3)=\{I\subset\{1,\dots,k\};a\le l_3 \text{ and } (I,J) \text{ is } l_1\text{-admissible}\}. \tag{5.3}$$

If $\min(l_3,p)=0$, $T^{(k)}(J;l_1,l_3)=\{\emptyset\}$. If $\min(l_3,p)>0$, we define the structure of colored graph on $T^{(k)}(J;l_1,l_3)$ as follows.

If $I\in T^{(k)}(J;l_1,l_3)$ and $I\ne I_{\max}(J)$, we draw an outgoing arrow from I. We denote the terminal of this arrow by $\xi(I)\in T^{(k)}(J,l_1,l_3)$ and associate the arrow with color $c(I)\in\{1,\dots,\min(l_3,p)\}$. The data $\xi(I)$ and $c(I)$ are determined as follows.

Consider $I=\{u_i\}$, $J=\{v_i\}$ and $\{v'_i\}$ as in Section 4.1. If $u_i=v_i$ for $1\le i\le a$, we have $a<\min(l_3,p)$ since otherwise $I=I_{\max}(J)$. We set $c(I)=a+1\le\min(l_3,p)$ and $\xi(I)=I\sqcup\{v'_{c(I)}-1\}$. Note that $v'_{c(I)}-1\notin I$ because $u_a=v_a<v_{c(I)}<v'_{c(I)}$. If there exists i such that $u_i>v_i$, we set $c(I)$ to be the minimal integer i satisfying this property, and $\xi(I)=\Bigl(I\backslash\{u_{c(I)}\}\Bigr)\sqcup\{u_{c(I)}-1\}$. Note that $u_{c(I)}-1\notin I$, since otherwise we have a contradiction

$$u_{c(I)}-1=u_{c(I)-1}=v_{c(I)-1}\le v_{c(I)}-1<u_{c(I)}-1.$$

We have

Lemma 5.1.1.

$$\bigsqcup_{I\in T^{(k)}(J;l_1,l_3)} R_{m,n}[l_1,l_2]_{I,J}=R_{m,n}[l_1,l_2,l_3]_J. \tag{5.4}$$

Proof. We use induction on l_3. If $l_3=0$, the statement is obvious because the union (5.4) is for a single element $I=\emptyset$. We reduce the proof for l_1,l_2,l_3,k to $l_1-1,l_2-1,l_3-1,k-1$.

Fix $J=\{v_1,\dots,v_b\}$ such that $b\le l_2$, and denote $R_I=R_{m,n}[l_1,l_2]_{I,J}$. We take the union of R_I over a maximal string $I[i]\in T^{(k)}(J,l_1,l_3)$ $(1\le i\le\gamma)$ of color 1:

$$I[1]\xrightarrow{1}I[2]\xrightarrow{1}\dots\xrightarrow{1}I[\gamma].$$

This is maximal in the sense that there is no arrow of color 1 pointing to $I[1]$ or from $I[\gamma]$. Each arrow of color 1 belongs to one and only one maximal string of color 1.

If $\#\left(I[\gamma]\right)=1$, $\gamma=v'_1-v_1+1$, $I[1]=\emptyset$ and $I[i]=\{v'_1-i+1\}$ for $2\le i\le\gamma$. If $a=\#\left(I[\gamma]\right)>1$, there exists a sequence

$$u_2<\dots<u_a$$

such that $\gamma=u_2-v_1$ and $I[i]=\{u_1[i],u_2,\dots,u_a\}$ where $u_1[i]=u_2-i$. Note that in the case $a=1$, the situation is the same if we set $u_2=v'_1+1$.

Consider the restriction $r[\alpha]\ge_{\varepsilon(I[i])_\alpha}\rho(I[i],J)_\alpha$ in $R_{I[i]}$ $(1\le i\le\gamma)$. Unless $v_1\le\alpha\le u_2-2$, $\varepsilon(I[i])_\alpha$ and $\rho(I[i],J)_\alpha$ are independent of i.

If $v_1\le\alpha\le u_2-2$, we have

$$\varepsilon(I[i])_\alpha \;=\; 1 \text{ if and only if } i=u_2-\alpha, \tag{5.5}$$

$$\rho(I[i],J)_\alpha \;=\; \begin{cases}\rho(I[1],J)_\alpha & \text{if } 1\le i\le u_2-\alpha-1;\\ \rho(I[1],J)_\alpha-1 & \text{if } u_2-\alpha\le i\le\gamma.\end{cases} \tag{5.6}$$

From these observations follows that $R_{I[i]}$ $(1\le i\le\gamma)$ are disjoint, and the union is characterized by the conditions that

$$r[\alpha]\ge_{\varepsilon(I[\gamma]\backslash\{v_1\})_\alpha}\rho(I[\gamma],J)_\alpha\ (1\le\alpha\le k),\quad s[\alpha]\ge_{\varepsilon(J)_\alpha}\sigma(J)_\alpha\ (1\le\alpha\le k). \tag{5.7}$$

Since $\rho(I[\gamma], J)_\alpha = 0$ for $1 \le \alpha \le v_1$, there is no restriction on $r[\alpha]$ for $1 \le \alpha \le v_1$. In particular, there is no restriction for $r[1]$.

Now, we modify the graph. We discard $I(i)$ $(1 \le i \le \gamma - 1)$ from $T^{(k)}(J, l_1, l_3)$ and replace the set $R_{I[\gamma]}$ by the union $R'_{I[\gamma]}$ characterized by (5.7). Carrying out this process for all the maximal strings of color 1, we obtain a new graph $T^{(k)}(J; l_1, l_3)'$ and the sets R'_I $(I \in T^{(k)}(J; l_1, l_3)')$. Observe that $I = \{u_1, \dots, u_{\#(I)}\} \in T^{(k)}(J; l_1, l_3)'$ satisfies the restriction $u_1 = v_1$ and there is no arrow of color 1 in $T^{(k)}(J; l_1, l_3)'$.

We see that the graph $T^{(k)}(J; l_1, l_3)'$ is isomorphic to $T^{(k-1)}(J'; l_1 - 1, l_3 - 1)$ where $J' = \{v_2 - 1, \dots, v_b - 1\}$. The isomorphism maps I to $I' = \{u_2 - 1, \dots, u_a - 1\}$ and identifies the color c in the former with the color $c - 1$ in the latter. We have $\rho_{l_1,k}(I, J)_\alpha = \rho_{l_1-1,k-1}(I', J')_{\alpha-1}$ for $2 \le \alpha \le k$, and $\varepsilon(I \backslash \{v_1\})_\alpha = 1$ if and only if $\varepsilon(I')_{\alpha-1} = 1$.

Therefore, the condition for $r[\alpha]$ in R'_I is exactly the same as the condition for $r[alpha - 1]$ in the subset $R_{m,n}[l_1 - 1, l_2 - 1, l_3 - 1]_{I',J'}$ at the level $k - 1$. Thus we have proved (5.4). ☐

5.2. **Union of $R_{m,n}[l_1, l_2, l_3]_J$ over J.** Consider the subsets indexed by J such that $\#(J) \le l_2$ (5.2):

$$R_{m,n}[l_1, l_2, l_3]_J \subset R_{m,n}.$$

They are disjoint. In fact, the restrictions on the riggings s given by $\sigma(J)$ and $\varepsilon(J)$ are disjoint (see Lemma 5.2.1 below).

The goal is to show that the union $R_{m,n}[l_1, l_2, l_3]_J$ over J is equal to $R_{m,n}[l_1, l_2, l_3]$.

If $l_3 = \min(l_1, l_2)$, we have

Lemma 5.2.1.

$$\bigsqcup_{J:\#(J) \le l_2} R_{m,n}[l_1, l_2, \min(l_1, l_2)]_J = R_{m,n}. \tag{5.8}$$

Proof. If $l_3 = \min(l_1, l_2)$, we have $\min(l_3, p) = p$ since $p \le \min(l_1, l_2)$. From this follows that $I_{\max}(J) = \{v_1, \dots, v_p\}$, and therefore, $\rho(I_{\max}(J), J) = 0$. Therefore, there is no restriction on $r[\alpha]$ in $R_{m,n}[l_1, l_2, \min(l_1, l_2)]_J$. We take the union of the riggings s subject to the restriction on $s[\alpha]$. This is equivalent to the special case of Lemma 5.1.1 where I, J, l_1, l_2, l_3 are replaced by $J, [1, l_2], k, l_2, l_2$, respectively. Therefore, the left hand side of (5.8) is disjoint and the equality holds. ☐

Set

$$\begin{aligned} C_1 &= \bigcup_{1 \le \alpha, \beta \le k} \bigcup_{0 \le i+j \le \tau^{(\alpha,\beta)}[l_1,l_2,l_3]-1} R^{(\alpha,\beta)}[i, j], &(5.9)\\ R^{(\alpha,\beta)}[i, j] &= \{(\mu, r; \nu, s) \in R_{m,n}; r[\alpha] = i, s[\beta] = j\}. &(5.10)\end{aligned}$$

It is easy to see that

$$C_1 = R_{m,n} \backslash R_{m,n}[l_1, l_2, l_3].$$

Set

$$\begin{aligned} C_2 &= R_{m,n} \backslash U, &(5.11)\\ U &= \bigsqcup_{J:\#(J) \le l_2} R_{m,n}[l_1, l_2, l_3]_J. &(5.12)\end{aligned}$$

Lemma 5.2.1 enables us to represent C_2, which is by definition the complement of union, as the union of complements. Namely, we have

$$C_2 = \bigsqcup_{J:\#(J)\leq l_2} R^c_J, \tag{5.13}$$

$$R^c_J = R_{m,n}[l_1,l_2,\min(l_1,l_2)]_J \backslash R_{m,n}[l_1,l_2,l_3]_J. \tag{5.14}$$

The goal is to show that

$$C_1 = C_2.$$

First we assume that $l_3 = 0$. In this case, we have $I_{\max}(J) = \emptyset$. We prove that (5.13) is equal to (5.9).

We call $K \subset [1,k]$ of the first kind if for some $\beta(K), b(K) \in \{1,\ldots,k\}$ it is of the form

$$K = [\beta(K) - b(K) + 1, \beta(K)]. \tag{5.15}$$

We will modify (5.13) and obtain another representation of the form

$$C_3 = \bigcup_{K:\text{ of the first kind and }\#(K)\leq l_2} R'^c_K, \tag{5.16}$$

$$R'^c_K = \bigcup_{\substack{1\leq\alpha\leq k\\ 0\leq i\leq\rho(\emptyset,K)_\alpha-1}} R'^c_K[\alpha,i], \tag{5.17}$$

$$R'^c_K[\alpha,i] = \{(\mu,r;\nu,s)\in R_{m,n};\ r[\alpha]=i, s[\beta(K)] = \sigma(K)_{\beta(K)}\}. \tag{5.18}$$

We start from a lemma on some property of the restriction (5.2) on the riggings s given by $\sigma(J) = \kappa[1,l_2] - \kappa(J)$ and $\varepsilon(J)$.

For $J \subset [1,k]$ such that $\#(J) \leq l_2$ we set

$$S_J = \{s = (s_1,\ldots,s_k) \in \mathbb{Z}^k_{\geq 0}; s_\alpha \geq_{\varepsilon(J)_\alpha} \sigma(J)_\alpha\ (1\leq\alpha\leq k)\}, \tag{5.19}$$

and for $K = [\beta - b + 1, \beta] \subset [1,k]$ such that $b \leq l_2$

$$S'_K = \{s \in \mathbb{Z}^k_{\geq 0}; s_\beta = \sigma(K)_\beta\}. \tag{5.20}$$

As we have already mentioned in the proof of Lemma 5.2.1, the subsets S_J are disjoint.

Lemma 5.2.2. *We have the inclusion*

$$S'_K \subset \cup_{J\geq K} S_J. \tag{5.21}$$

Proof. We will prove this by induction on K with respect to the ordering defined in Section 4.1. We see that the statement is true for the maximal element $K = [1,l_2]$. In fact, if $K = [1,l_2]$ the statement $S'_K = S_K$ follows from $\sigma(K) = 0$ and $\varepsilon([1,l_2])_\alpha = 1$ if and only if $\alpha = l_2$. This is the base of the induction.

Now assume that the statement is true for all K' of the first kind such that $K' > K$. We will show that there exists a subset $\overline{S}_K$ satisfying

$$\overline{S}_K \subset \bigsqcup_{J \geq K} S_J, \tag{5.22}$$

$$S'_K \backslash \overline{S}_K \subset \bigcup_{\substack{K' > K \\ K' \text{: of the first kind}}} S'_{K'}, \tag{5.23}$$

This will close the induction steps.

We fix $K = [\beta - b + 1, \beta]$ and define

$$\overline{S}_K = \bigsqcup_{\substack{\beta \in J \geq K \\ \#(J \cap [1, \beta-1]) = b-1}} S_J. \tag{5.24}$$

Namely, we take the disjoint union over $J = \{v_1, \ldots, v_{b-1}, \beta, v_{b+1}, \ldots, v_{b'}\}$ such that $1 \leq v_1 < \cdots < v_{b-1} < \beta < v_{b+1} < \cdots < v_{b'}$ with $b' \leq l_2$. Note that the element β is fixed, $v_1, \ldots, v_{b-1}$ move around the interval $[1, \beta - 1]$ and new elements $v_{b+1}, \ldots, v_{b'}$ are added in the interval $[\beta + 1, k]$. We have (5.22) obviously.

By the same argument as in the proof of Lemma 5.1.1 we obtain

$$\overline{S}_K = \{s \in \mathbb{Z}^k_{\geq 0}; s_\alpha \geq \sigma(K_{\max})_\alpha \ (1 \leq \alpha \leq k)\}, \tag{5.25}$$

where $K_{\max} = [1, b-1] \sqcup [\beta, \beta + l_2 - b]$.

We have the following values of $\sigma(K)_\alpha$ and $\sigma(K_{\max})_\alpha$.

If $\beta \geq l_2$, then

$$\sigma(K)_\alpha = \begin{cases} \alpha & (1 \leq \alpha \leq l_2); \\ l_2 & (l_2 \leq \alpha \leq \beta - b); \\ l_2 + \beta - \alpha - b & (\beta - b \leq \alpha \leq \beta); \\ l_2 - b & (\beta \leq \alpha \leq k), \end{cases} \tag{5.26}$$

$$\sigma(K_{\max})_\alpha = \begin{cases} \max(0, \alpha - b + 1) & (1 \leq \alpha \leq l_2); \\ l_2 - b + 1 & (l_2 \leq \alpha \leq \beta - 1); \\ \max(0, l_2 + \beta - \alpha - b) & (\beta - 1 \leq \alpha \leq k). \end{cases} \tag{5.27}$$

If $\beta \leq l_2$, then

$$\sigma(K)_\alpha = \begin{cases} \alpha & (1 \leq \alpha \leq \beta - b); \\ \beta - b & (\beta - b \leq \alpha \leq \beta); \\ \alpha - b & (\beta \leq \alpha \leq l_2); \\ l_2 - b & (l_2 \leq \alpha \leq k), \end{cases} \tag{5.28}$$

$$\sigma(K_{\max})_\alpha = \begin{cases} \max(0, \alpha - b + 1) & (1 \leq \alpha \leq \beta - 1); \\ \beta - b & (\beta - 1 \leq \alpha \leq l_2); \\ \max(0, l_2 + \beta - \alpha - b) & (l_2 \leq \alpha \leq k). \end{cases} \tag{5.29}$$

Now we will prove (5.23). We have $\sigma(K)_\beta = \sigma(K_{\max})_\beta$. Therefore,

$$S'_K \backslash \overline{S}_K = \bigcup_{\substack{\alpha\neq\beta \\ 0\le i\le \sigma(K_{\max})_\alpha -1}} \{s\in \mathbb{Z}^k_{\geq 0}; s_\alpha = i\} \cap S'_K. \tag{5.30}$$

We take K' in (5.23) to be $K_{\alpha,b'} = [\alpha - b' + 1, \alpha]$. We have $K_{\alpha,b'} \subset [1,k]$ and $K_{\alpha,b'} \geq K$ if and only if

$$\max(b, b+\alpha-\beta) \le b' \le \min(l_2,\alpha). \tag{5.31}$$

By case checking one can prove that the set of integers consisting of the values of $\sigma(K_{\alpha,b'})_\alpha = \min(l_2,\alpha) - b'$ where b' runs over (5.31), contains $[0, \sigma(K_{\max})_\alpha - 1]$ appearing in (5.30). For example, if $1 \le \alpha \le l_2 \le \beta$, we have $\sigma(K_{\max})_\alpha = \max(0, \alpha - b + 1)$ and $\sigma(K_{\alpha,b'})_\alpha = \alpha - b'$. Therefore, we obtain

$$\cup_{0\le i\le \sigma(K_{\max})_\alpha - 1}\{s\in\mathbb{Z}^k_{\geq 0}; s_\alpha = i\}\cap S'_K \subset \cup_{b\le b'\le \alpha} S'_{K_{\alpha,b'}}.$$

Other cases are similar. ∎

Now we prove

Lemma 5.2.3. *Assume that $l_3 = 0$. We have*

$$C_2 = C_3.$$

Proof. If $l_3 = 0$ we have

$$\begin{aligned} R^c_J &= \bigcup_{\substack{1\le\alpha\le k \\ 0\le i\le \rho(\emptyset,J)_\alpha - 1}} R^c_J[\alpha,i], \\ R^c_J[\alpha,i] &= \{(\mu,r;\nu,s)\in R_{m,n}; \\ &\qquad r[\alpha] = i, s[\beta] \geq_{\varepsilon(J)_\beta} \sigma(J)_\beta \ (1\le\beta\le k)\}. \end{aligned} \tag{5.32}$$

First we show that if K is an interval of the first kind (5.15) we have

$$R'^c_K \subset \cup_{J\geq K} R^c_J. \tag{5.33}$$

¿From this follows that $C_3 \subset C_2$.

If $J' > J$ then $\rho(\emptyset, J') \geq \rho(\emptyset, J)$. Therefore, in order to show (5.33) one can forget the restriction on r. Then, it follows from Lemma 5.2.2.

To finish the proof, we show that $C_2 \subset C_3$. Consider $J = J^{(1)} \sqcup \cdots \sqcup J^{(h)}$ where

$$J^{(j)} = [\beta^{(j)} - b^{(j)} + 1, \beta^{(j)}]$$

and $\beta^{(j)} < \beta^{(j+1)} - b^{(j+1)}$. Set

$$K^{(j)} = [\beta^{(j)} - (b(1) + \cdots + b(j)) + 1, \beta^{(j)}] \quad (1\le j\le h).$$

We will show that

$$R^c_J \subset \cup_{j=1}^h R'^c_{K^{(j)}}. \tag{5.34}$$

Note that $\sigma(J)_{\beta^{(j)}} = \sigma(K^{(j)})_{\beta^{(j)}}$ and $\varepsilon(J)_{\beta^{(j)}} = 1$. Therefore, the condition for the riggings s in $R^c_J[\alpha,i]$ is stronger than $R'^c_{K^{(j)}}[\alpha,i]$. Namely, we have $R^c_J[\alpha,i] \subset R'^c_{K^{(j)}}[\alpha,i]$.

For any α we can find j such that

$$\beta^{(j)} - b^{(j)} + 1 \le \alpha \le \beta^{(j+1)} - b^{(j+1)}.$$

Then we have

$$\rho(\emptyset, J)_\alpha = \rho(\emptyset, K^{(j)})_\alpha.$$

The statement (5.34) follows from this. □

Next we show

Lemma 5.2.4. *Assume that $l_3 = 0$. We have*

$$C_1 = C_3.$$

Proof. We show that (5.16) is equal to (5.9) by case checking for each case of the ordering of α, β, l_1, l_2. There are 24 cases. Here we give the details for the case $l_2 \leq l_1 < \alpha \leq \beta$. Other cases are similar.

If $l_2 \leq l_1 < \alpha \leq \beta$ the intervals K which appear in (5.16) satisfying $\beta(K) = \beta$ are of the form $J = [\gamma, \beta]$ where $\beta - l_2 + 1 \leq \gamma \leq \alpha$. For such K we have

$$\rho(\emptyset, K)_\alpha = l_1 - \gamma + 1, \quad \sigma(K)_\beta = l_2 - (\beta - \gamma + 1).$$

Therefore, the pair of integers $(\rho, \sigma) = (\rho(\emptyset, K)_\alpha - 1, \sigma(K)_\beta)$ runs over the set $\{(\rho, \sigma); \rho, \sigma \geq 0, \rho + \sigma = l_1 + l_2 - \beta - 1\}$. On the other hand we have

$$\tau^{(\alpha,\beta)}[l_1, l_2, 0] = \min(\alpha, \beta, l_1, l_2, l_1 + \beta - \alpha, , l_2 + \alpha - \beta, l_1 + l_2 - \alpha, l_1 + l_2 - \beta) = l_1 + l_2 - \beta.$$

This completes the proof. □

Finally, we have

Lemma 5.2.5.

$$\bigsqcup_{J : \#(J) \leq l_2} R_{m,n}[l_1, l_2, l_3]_J = R_{m,n}[l_1, l_2, l_3]. \tag{5.35}$$

Proof. By Lemmas 5.2.3 and 5.2.4 we have shown (5.35) for $l_3 = 0$. Let us reduce the proof to the case $l_3 = 0$. Suppose that $l_3 > 0$. Then, we have $l_1, l_2 > 0$. We will reduce this case to the case where l_1, l_2, l_3 and k replaced by $l_1 - 1, l_2 - 1, l_3 - 1$ and $k - 1$, respectively.

Note that the union is taken over J such that $\#(J) \leq l_2$, i.e., $J \in T^{(k)}([1, l_2]; k, l_2)$. Therefore, we refer to the structure of colored graph in this set.

Recall the definition of $\rho_{\max}(J)$ given by (5.1). If J varies on a maximal string of color 1, then only v_1 changes. However, we see that the $\rho_{\max}(J)$ is independent of v_1 because for $I = I_{\max}(J)$ we have $u_1 = v_1$. It follows that the vector $\rho_{\max}(J)$ is constant on the maximal string. Therefore, we can take the union over J on maximal strings of color 1 only on the riggings s forgetting r.

Taking unions over all of the maximal strings of color 1, we can rewrite the left hand side of (5.35) as the union of the resulting subsets over such J that satisfies $1 \in J$, i.e., of the form $J = \{1, v_2, \ldots, v_b\}$. The subgraph of $T^{(k)}([1, l_2]; k, l_2)$ consisting of such J is isomorphic to $T^{(k-1)}([1, l_2 - 1]; k - 1, l_2 - 1)$ by mapping J to $J' = \{v_2 - 1, \ldots, v_b - 1\}$ and identifying the color c in the former with the color $c - 1$ in the latter.

We have

$$\rho_{l_1,k}(I_{\max}(J), J)_\alpha = \rho_{l_1-1,k-1}(I_{\max}(J'), J')_{\alpha-1} \quad (2 \leq \alpha \leq k).$$

Therefore, we have

$$\begin{aligned}\sigma_{l_2,k}(J)_\alpha &= (\kappa([1,l_2]) - \kappa(J))_\alpha \\ &= (\kappa([1,l_2-1]) - \kappa(J'))_{\alpha-1} \\ &= \sigma_{l_2-1,k-1}(J')_{\alpha-1}.\end{aligned}$$

Note also that

$$\tau^{(\alpha,\beta)}[l_1,l_2,l_3] = \tau^{(\alpha-1,\beta-1)}[l_1-1,l_2-1,l_3-1].$$

Thus, we have reduced the case l_1, l_2, l_3, k to $l_1-1, l_2-1, l_3-1, k-1$. □

In conclusion, we have

Proposition 5.2.6.

$$R_{m,n}[l_1,l_2,l_3] = \bigsqcup_{\#(I)\le l_3, \#(J)\le l_2} R_{m,n}[l_1,l_2]_{I,J}. \tag{5.36}$$

Proposition 5.2.7.

$$R^{(M,N)}_{m,n}[l_1,l_2,l_3] = \bigsqcup_{\#(I)\le l_3, \#(J)\le l_2} R^{(M,N)}_{m,n}[l_1,l_2]_{I,J}. \tag{5.37}$$

6. Decomposition of $R^{(M,N-1)}_{m-a,n-b}[l'_1,l'_2,l'_3]$

Fix k, l_1, a, b such that $0 \le a \le l_1 \le k$, $a \le b = a+c \le k$, and define l'_1, l'_2, l'_3 by (2.16). We denote $m' = m-a$ and $n' = n-b$ in this section.

The aim of this section is to decompose the set $R^{(M,N-1)}_{m',n'}[l'_1,l'_2,l'_3]$ as

$$R^{(M,N-1)}_{m',n'}[l'_1,l'_2,l'_3] = \bigsqcup_{\#(I)=a, \#(J)=b} R^{(M,N-1)}_{m',n'}[l_1]^{I,J}. \tag{6.1}$$

Again, it is enough to decompose $R_{m',n'}[l'_1,l'_2,l'_3]$ as

$$R_{m',n'}[l'_1,l'_2,l'_3] = \bigsqcup_{\#(I)=a, \#(J)=b} R_{m',n'}[l_1]^{I,J}. \tag{6.2}$$

It is useful to note that if $\alpha = k$ or $\beta = k$ then $\tau^{(\alpha,\beta)}[l'_1,l'_2,l'_3] = 0$. Also, because of (4.44) there are no restrictions on $r'[k]$ nor $s'[k]$ in the definition (4.48) of the upper subsets $R_{m',n'}[l_1]^{I,J}$. Therefore, we can restrict our discussion on k vectors in this section to the interval $1 \le \alpha \le k-1$.

The proof is divided into two cases: $l_1 + c \ge k$ and $l_1 + c < k$.

6.1. Case $l_1 + c \ge k$.

In this case, we have (see (2.16))

$$l'_1 = k-a, \quad l'_2 = k-c \quad l'_3 = k-b.$$

and

$$\begin{aligned}\sigma'(I,J) &= \kappa(J) - \kappa(I) - \kappa[k-c+1,k], \\ \rho'(I,J) &= \kappa(I) - \kappa[k-a+1,k].\end{aligned}$$

Note, in particular, that the l_1-dependence disappears. We write $R^{I,J}_{m',n'}$ for $R_{m',n'}[l_1]^{I,J}$.

First, we fix the subset $I=\{u_1,\dots,u_a\}$ and take the union over $J=\{v_1,\dots,v_b\}$. This is similar to Lemma 5.1.1 We use (4.15) for $J_{\min}$, (4.42) with $\tilde I=I$ for $\rho'(I)=\rho'(I,J_{\min})$ and (4.45) for $\sigma'(I)=\sigma'(I,J_{\min})$. They are all independent of l_1.

Set

$$\begin{aligned} R^I_{m',n'} &= \{(\mu',r';\nu',s')\in R_{m',n'}; \\ &\quad r'[\alpha]\geq^{\varepsilon(I)_\alpha}\rho'(I)_\alpha(1\le\alpha\le k-1), s'[\alpha]\ge\sigma'(I)_\alpha(1\le\alpha\le k-1)\}. \end{aligned} \tag{6.3}$$

Lemma 6.1.1. *For $I=\{u_1,\dots,u_a\}$, we have*

$$\bigsqcup_{\substack{J=\{v_1,\dots,v_b\}\subset\{1,\dots,k\}\\ v_1\le u_1,\dots,v_a\le u_a}} R^{I,J}_{m',n'}=R^I_{m',n'}.$$

Proof. The proof of Lemma 6.1.1 is parallel to that of Lemma 5.1.1.

In Lemma 6.1.1 the restriction on r' in $R^{I,J}_{m',n'}$ is independent of J and the restriction on s' is of the form

$$\begin{aligned} s'[\alpha]&\ge(\kappa(J)+A)_\alpha &&\text{if } \varepsilon(J)_\alpha\ne-1,\\ s'[\alpha]&=(\kappa(J)+A)_\alpha &&\text{if } \varepsilon(J)_\alpha=-1. \end{aligned}$$

Here A is a k-vector independent of J.

In Lemma 5.1.1 the restriction on s in $R_{m,n}[l_1,l_2]_{I,J}$ is independent of I and the restriction on r is of the form

$$\begin{aligned} r[\alpha]&\ge(-\kappa(I)+B)_\alpha &&\text{if } \varepsilon(I)_\alpha\ne1,\\ r[\alpha]&=(-\kappa(I)+B)_\alpha &&\text{if } \varepsilon(I)_\alpha=1. \end{aligned}$$

Here B is a k-vector independent of I.

In Lemma 6.1.1 the union is taken over J such that

$$J_{\min}\le J\le[1,b],$$

where $J_{\min}$ is given by (4.15), and in Lemma 5.1.1 the union is taken over I such that

$$\emptyset\le I\le I_{\max},$$

where $I_{\max}$ is given by (5.1).

Recall that in the proof of Lemma 5.1.1 we take the union over the maximal strings of color 1 as the first inductive step. Similarly, in the setting of Lemma 6.1.1 we take the union over strings $J[i]$ $(1\le i\le\gamma)$ of the form $J[i]=\{v_1,\dots,b_{b-1}[i]\}$ $(v_1<\dots<v_{b-1})$ and $v_b[i]=v_{b-1}+i$. Here

$$\gamma=\begin{cases} k-v_{b-1} & \text{if } b>a;\\ u_a-v_{b-1} & \text{if } b=a.\end{cases}$$

The difference between two cases is that $\#(J)=b$ is fixed in Lemma 6.1.1, while $\#(I)$ varies in Lemma 5.1.1. However, if we consider $\overline{I}=I\sqcup\{v'_p,\dots,v'_{\#(I)+1}\}$ instead of I, $\#(\overline{I})=p$ is fixed and two cases are completely parallel.

Therefore, the union is obtained by substituting J by $J_{\min}$ and make the restriction on s' unmarked. $\square$

Now we translate the formula (to be proved)

$$\bigsqcup_{\#(I)=a} R^I_{m',n'} = R_{m',n'}[l'_1, l'_2, l'_3] \tag{6.4}$$

into the formula

$$R_{m',n'}[l_1, l_2, 0] = \bigsqcup_{\#(J)\le l_2} R_{m',n'}[l_1, l_2]_{\emptyset,J}, \tag{6.5}$$

which is the special case of (5.36) with $l_3 = 0$. We use the case

$$l_1 = c \text{ and } l_2 = a$$

by the following reason.

In $R^I_{m',n'}$ we have

$$\begin{aligned} \rho'(I) &= \kappa(I) - \kappa[k-a+1, k] = \kappa(I) - \kappa[l'_1 + 1, k], &(6.6)\\ \sigma'(I) &= \Big(\kappa[k-b+1, k-c] - \kappa(I)\Big)^+ = \Big(\kappa[l'_3 + 1, l'_2] - \kappa(I)\Big)^+. &(6.7)\end{aligned}$$

On the other hand, in (6.5) we have

$$\begin{aligned} \rho(\emptyset, J) &= \Big(\kappa(J) - \kappa[c+1, c+\#(J)\Big)^+,\\ \sigma(J) &= \kappa[1, a] - \kappa(J). \end{aligned}$$

Note, in particular, that $\rho(\emptyset, J)_k = \sigma(J)_k = 0$.

We define an involution of the set $\{1, \dots, k\}$ by

$$i^\dagger = k + 1 - i.$$

Using

$$\kappa(i)_\alpha + \kappa(i^\dagger)_{k-\alpha} = 1, \tag{6.8}$$

we obtain

$$\rho'(I)_\alpha = \sigma(I^\dagger)_{k-\alpha}, \tag{6.9}$$

$$\sigma'(I)_\alpha = \rho(\emptyset, I^\dagger)_{k-\alpha}. \tag{6.10}$$

In this way, we can translate (6.4) into (6.5) except that the union is taken over I with the fixed size $\#(I) = a$ in (6.4) while J in (6.5) is only restricted by $\#(J) \le a$.

Therefore, we need some modification. We will take the union in (6.5) partially so that only J of size a remain.

Given $J = \{v_1, \dots, v_{a'}\}$ such that $a' \le a$ we define the closure of J by

$$\overline{J} = J \sqcup \{w_1, \dots, w_{a-a'}\} \tag{6.11}$$

where $w_1 < \cdots < w_{a-a'}$ are chosen to be the maximal $a - a'$ elements in $[1, k]\backslash J$.

For K such that $\#(K) = a$ we set

$$\begin{aligned} \big(R_{m',n'}\big)_K = \{&(\mu, r; \nu, s) \in R_{m',n'};\\ &r[\alpha] \ge \rho(\emptyset, K)_\alpha \quad (1 \le \alpha \le k-1),\\ &s[\alpha] \ge_{\varepsilon(K)} \sigma(K)_\alpha \quad (1 \le \alpha \le k-1)\}. \end{aligned} \tag{6.12}$$

Note that we impose no restrictions at $\alpha = k$.

We have

Lemma 6.1.2.

$$\big(R_{m',n'}\big)_K = \bigsqcup_{J:\overline{J}=K} R_{m',n'}[c,a]_{\emptyset,J}. \tag{6.13}$$

Proof. Suppose that $\#(J) = a' \le a$ and $\overline{J} = K = K_0 \sqcup [k-d+1,k]$ where $k-d \notin K_0$. Then, we have $K_0 \subset J$, and K_0 and d are uniquely determined from K. We have

$$\begin{aligned}\rho(\emptyset,J) &= (\kappa(J) - \kappa[c+1,c+a'])^+\\ &= (\kappa(K_0) - \kappa[c+1,c+a-d] + \kappa(J\backslash K_0) - \kappa[c+a-d+1,c+a'])^+.\end{aligned}$$

Note that $\#(K_0) = a-d$ and $\#(J\backslash K_0) = a'+d-a$. Therefore, we have $(\kappa(K_0)-\kappa[c+1,c+a-d])_\alpha \le 0$ for $\alpha \ge c+a-d$. Since $J\backslash K_0 \subset [k-d+1,k]$, we have

$$\kappa(J\backslash K_0) \le \kappa[k-d+1,k+a'-a] \le \kappa[c+a-d+1,c+a'].$$

¿From these observations follows that

$$\begin{aligned}\rho(\emptyset,J) &= (\kappa(K_0) - \kappa[c+1,c+a-d])^+\\ &= \rho(\emptyset,K).\end{aligned}$$

The set J satisfies $\#(J) \le a$ and $\overline{J} = K$ if and only if $J = K_0 \sqcup J'$ where $J' \subset [k-d+1,k]$.

By a similar argument as the proof of Lemma 5.1.1, taking the union of $R_{m',n'}[c,a]_{\emptyset,J}$ over J', we obtain $(R_{m',n'})_K$. □

Now we are ready to finish the proof of (6.2) for $l_1 + c \ge k$.

Lemma 6.1.3.

$$\bigsqcup_{\#(I)=a} R^I_{m',n'} = R_{m',n'}[l'_1,l'_2,l'_3] \tag{6.14}$$

Proof. Let us observe that there is a correspondence between $R^I_{m',n'}$ and $(R_{m',n'})_K$. We have (6.9) and (6.10). Moreover, because of

$$\varepsilon(i)_\alpha + \varepsilon(k+1-i)_{k-\alpha} = 0, \tag{6.15}$$

$\varepsilon(I^\dagger)_{k-\alpha} = 1$ if and only if $\varepsilon(I)_\alpha = -1$.

Finally, note that

$$\tau^{(k-\beta,k-\alpha)}[k-a,k-c,k-a-c] = \tau^{(\alpha,\beta)}[c,a,0].$$

In this way, (6.14) follows from Proposition 5.2.6. □

6.2. **Case $l_1 + c < k$.** In this case, we have (see (2.16))

$$l'_1 = l_1 + c - a, \quad l'_2 = k - c \quad l'_3 = l_1 - a.$$

We use (4.16) for $J_{\min}$, (4.42) for $\rho'(I) = \rho'(I,J_{\min})$ and (4.43) for $\sigma'(I) = \sigma'(I,J_{\min})$. Namely, we have

$$\rho'(\tilde{I}) = \kappa(\tilde{I}) - \kappa[l'_1+1,k], \tag{6.16}$$

$$\sigma'(\tilde{I}) = (\kappa[l'_3+1,l'_2] - \kappa(\tilde{I}))^+. \tag{6.17}$$

Because $I' \subset [l_1+1,k]$, we have

$$(\kappa[l_1-a+1,l_2'] - \kappa(\tilde I))^+ = (\kappa[l_1-a+1,l_1] - \kappa(I))^+ + \kappa[l_1+1,l_2'] - \kappa(I'). \tag{6.18}$$

In the following lemma, when $l_1+c<k$, we define $R^{\tilde I}_{m',n'}$ differently from $R^{I}_{m',n'}$ defined in (6.3) when $l_1+c \ge k$.

Lemma 6.2.1. *Suppose that $\tilde I$ satisfies the condition* (4.11). *Set*

$$\begin{aligned}\tilde R^{\tilde I}_{m',n'} \quad = \quad & \{(\mu',r';\nu',s') \in R_{m',n'};\\ & r'[\alpha] \ge^{\varepsilon(I)_\alpha} \rho'(\tilde I)_\alpha (1\le\alpha\le k-1), s'[\alpha] \ge_{\varepsilon(I')_\alpha} \sigma'(\tilde I)_\alpha (1\le\alpha\le k-1)\}.\end{aligned} \tag{6.19}$$

We have

$$\bigsqcup_{\tilde J} R_{m',n'}[l_1]^{\mathfrak{c}(\tilde I,\tilde J)} = \tilde R^{\tilde I}_{m',n'}$$

where the summation in the LHS is taken over all $\tilde J$ satisfying (4.12) (see (4.13) for $(I,J) = \mathfrak{c}(\tilde I,\tilde J)$).

Proof. The proof of this lemma is parallel to the proof of Lemma 6.1.1. Note that $\rho'(\tilde I)$ (see (4.42)) does not depend on $\tilde J$. The summation extends to all $\tilde J$ satisfying

$$[1,a+u_{a+1}-l_1-1] \ge \tilde J \ge \tilde J_{\min}$$

where

$$\tilde J_{\min} = \{\min(u_i, l_1-a+i)\}_{1\le i\le a} \sqcup [l_1+1,u_{a+1}-1].$$

The restriction on $s[\alpha]$ in $R_{m',n'}[l_1]^{\mathfrak{c}(\tilde I,\tilde J)}$ is given by $\sigma'(I,J)_\alpha$. It is marked if and only if $\varepsilon(J)_\alpha = -1$. The restriction on $s[\alpha]$ in the union over $\tilde J$ is given by $\sigma'(\tilde I)_\alpha$. The marking will change as follows. Because of $J\cap[1,u_{a+1}-1] = \tilde J$ the restriction on $s[\alpha]$ is unmarked if $\alpha \in [1,u_{a+1}-1]$. In the interval $\alpha\in[u_{a+1},k]$, the marking is unchanged because $J' = J\cap[u_{a+1},k]$ is fixed.

Decompose the interval $[u_{a+1},k] = I'\sqcup J'$ into subintervals $I'_1,\dots,I'_h$ constituting I' and $J'_1,\dots,J'_h$ constituting J' in such a way that $\max(I'_i)+1 = \min(J'_i)$ and $\max(J'_i)+1=\min(I'_{i+1})$. Note that $u_{a+1}\in I'_1$ and J'_h may be empty. The restriction on $s[\alpha]$ is marked if and only if $\alpha = \min(J'_i)-1$ for some $1\le i\le h$. This is equivalent to say that it is marked if and only if $\alpha=\max(I'_i)$ for some $1\le i\le h$ except for $\alpha = k$. Therefore, we have the marking given in (6.19). ☐

We remake the union over $\tilde I$ as follows.

Lemma 6.2.2. *Set*

$$\begin{aligned}\overline{R}^{\tilde I}_{m'.n'} = \{&(\mu',r';\nu',s') \in R_{m',n'};\\ & r'[\alpha] \ge^{\varepsilon(\tilde I)_\alpha} \rho'(\tilde I)_\alpha \quad (1\le\alpha\le k-1; \alpha\ne l_1 \textit{ if } u_{a+1}=l_1+1),\\ & r'[l_1] \ge \rho'(\tilde I)_{l_1} \textit{ if } u_{a+1} = l_1+1,\\ & s'[\alpha] \ge \sigma'(\tilde I)_\alpha \quad (1\le\alpha\le k-1)\}.\end{aligned} \tag{6.20}$$

We have

$$\bigsqcup_{\tilde I} \tilde R^{\tilde I}_{m',n'} = \bigsqcup_{\tilde I} \overline{R}^{\tilde I}_{m',n'}. \tag{6.21}$$

Proof. Fix $\tilde{I}$. First, we can rewrite each subset $\tilde{R}^{\tilde{I}}_{m',n'}$ in terms of subsets which have the same restriction on r' as $\tilde{R}^{\tilde{I}}_{m',n'}$ but a different unmarked restriction on s'. If the restriction is $s[\alpha]=\sigma'_\alpha$, we rewrite it as the difference of two unmarked restrictions $s[\alpha]\geq\sigma'_\alpha$ and $s[\alpha]\geq\sigma'_\alpha+1$. We extend this argument to all α with marked conditions by using the inclusion-exclusion principle, which will be explained below.

Let

$$I'=I'_1\sqcup\cdots\sqcup I'_h$$

be the decomposition considered in the proof of Lemma 6.2.1. Denote by $\tilde{I}_{\rm right}$ the set of integers $\max(I'_i)$ for $1\leq i\leq h$ except when it is equal to k. This is exactly the set of α such that the restriction on $s'[\alpha]$ is marked in $\tilde{R}^{\tilde{I}}_{m',n'}$. For each $K\subset\tilde{I}_{\rm right}$ we denote by $\tilde{I}_K$ the subset obtained from $\tilde{I}$ by replacing all the elements $u\in K$ with $u+1$. Because of (6.18) we have

$$\sigma'(\tilde{I}_K)_\alpha=\begin{cases}\sigma'(\tilde{I})_\alpha+1 & \text{if } \alpha\in K\subset[l_1+1,k];\\ \sigma'(\tilde{I})_\alpha & \text{otherwise.}\end{cases}$$

We set

$$\begin{aligned}R^{\tilde{I},\tilde{I}_K}_{m'.n'} \quad = \quad & \{(\mu',r',\nu',s')\in R_{m',n'};\\ & r'[\alpha]\geq^{\varepsilon(\tilde{I})_\alpha}\rho'(\tilde{I})_\alpha \quad (1\leq\alpha\leq k-1),\\ & s'[\alpha]\geq\sigma'(\tilde{I}_K)_\alpha \quad (1\leq\alpha\leq k-1)\}.\end{aligned} \tag{6.22}$$

For a subset $R\subset R_{m',n'}$ we denote by $1_{{\rm supp}(R)}$ its support function, i.e., $1_{{\rm supp}(R)}(x)=1$ if $x\in R$ and $1_{{\rm supp}(R)}(x)=0$ if $x\notin R$. The inclusion-exclusion principle tells us that

$$1_{{\rm supp}(\tilde{R}^{\tilde{I}}_{m',n'})}=\sum_{K\subset\tilde{I}_{\rm right}}(-1)^{\#(K)}1_{{\rm supp}(R^{\tilde{I},\tilde{I}_K}_{m',n'})}.$$

Thus we obtain

$$1_{{\rm supp}(\bigsqcup_{\tilde{I}}\tilde{R}^{\tilde{I}}_{m',n'})}=\sum_{\tilde{I}}\sum_{K\subset\tilde{I}_{\rm right}}(-1)^{\#(K)}1_{{\rm supp}(R^{\tilde{I},\tilde{I}_K}_{m',n'})}. \tag{6.23}$$

Denote by $\tilde{I}_{\rm left}$ the set of integers $\min(I'_i)$ for $1\leq i\leq h$ except when it is equal to l_1+1. This is exactly the set of $\alpha\in[u_{a+1},k]$ such that the restriction on $r'[\alpha-1]$ is marked in $\overline{R}^{\tilde{I}}_{m',n'}$. For each $K\subset\tilde{I}_{\rm left}$ we denote by $\tilde{I}^K$ the subset obtained from $\tilde{I}$ by replacing all the elements $u\in K$ with $u-1$. Because of (6.16) we have

$$\rho'(\tilde{I}^K)_\alpha=\begin{cases}\rho'(\tilde{I})_\alpha+1 & \text{if } \alpha+1\in K\subset[l_1+1,k];\\ \rho'(\tilde{I})_\alpha & \text{otherwise.}\end{cases}$$

Denote by $\tilde{\mathfrak{I}}$ the set of $\tilde{I}$ satisfying (4.11). Note that

$$\{(\tilde{I},\tilde{I}_K);\tilde{I}\in\tilde{\mathfrak{I}},K\in\tilde{I}_{\rm right}\} \quad = \quad \{(\tilde{I}^K,\tilde{I});\tilde{I}\in\tilde{\mathfrak{I}},K\in\tilde{I}_{\rm left}\}. \tag{6.24}$$

Therefore, we have

$$(6.23)=\sum_{\tilde{I}}\sum_{K\subset\tilde{I}_{\rm left}}(-1)^{\#(K)}1_{{\rm supp}(R^{\tilde{I}^K,\tilde{I}}_{m',n'})}. \tag{6.25}$$

The inclusion-exclusion principle again tells us that

$$\sum_{K\subset\tilde I_{\rm left}}(-1)^{\#(K)}1_{{\rm supp}(R^{\tilde I^K,\tilde I}_{m',n'})}=1_{{\rm supp}(\overline{R}^{\tilde I}_{m',n'})}$$

Therefore, we obtain (6.21). □

Now we finish the case $l_1+c<k$.

Set $\tilde a=a+k-l_1-c$. For $I\subset\{1,\dots,k\}$ such that $I=\{u_1,\dots,u_{\tilde a}\}$ $(u_1<\cdots<u_{\tilde a})$ we define the closure of I by

$$\overline{I}=I\backslash\{u_{a+1},\dots,u_{a+d}\}\sqcup[l_1+1,l_1+d] \tag{6.26}$$

where

$$d=\begin{cases}0 & \text{if } u_{a+1}>l_1+1;\\ \max\{i;u_{a+i}\le l_1+i\} & \text{if } u_{a+1}\le l_1+1.\end{cases} \tag{6.27}$$

Note that if $d\neq0$ and $\alpha\le l_1+d$, then

$$\sigma'(I)_\alpha=\big((\kappa[l_1-a+1.k-c]-\kappa(I)\big)^+_\alpha=0.$$

Therefore, we have

$$\sigma'(I)=\sigma'(\overline{I}). \tag{6.28}$$

The set I satisfies the condition $u_{a+1}\ge l_1+1$ if and only if $\overline{I}=I$.

Lemma 6.2.3. *Suppose that $\tilde I$ satisfies (4.11). Consider I with its closure equal to $\tilde I$. The subset $R^I_{m',n'}$ is defined by (6.3), in which the definitions (6.6) for $\rho'(I)$ and (6.7) for $\sigma'(I)$ are used. We use l'_1,l'_2,l'_3 given by (6.1) with a replaced by $\tilde a$. Then, the formulas for l'_1,l'_2,l'_3, $\rho'(I)$ and $\sigma'(I)$ become exactly equal to those used in $\tilde R^{\tilde I}_{m',n'}$. With this understanding, we have*

$$\bigsqcup_{I:\overline{I}=\tilde I}R^I_{m',n'}=\tilde R^{\tilde I}_{m',n'}. \tag{6.29}$$

Proof. Suppose that d is given by (6.27) for $\tilde I=\{u_1,\dots,u_{\tilde a}\}$. If $d=0$, then $\overline{I}=\tilde I$ implies $I=\tilde I$, and (6.29) is obvious. If $d\ge1$, the union in the left hand side is over I such that

$$I=\tilde I\backslash[l_1+1,l_1+d]\sqcup I'$$

where

$$[u_a+1,u_a+d]\ge I'\ge[l_1+1,l_1+d].$$

Since $\sigma'(I)=\sigma'(\tilde I)$, by a similar argument as the proof of Lemma 5.1.1, we take the union of $R^I_{m',n'}$ and obtain $\tilde R^{\tilde I}_{m',n'}$. □

Lemma 6.2.4. *We have*

$$\bigsqcup_{\substack{\#(\tilde I)=a+k-l_1-c\\ u_{a+1}\ge l_1+1}}\tilde R^{\tilde I}_{m',n'}=R_{m',n'}[l'_1,l'_2,l'_3]. \tag{6.30}$$

Proof. This is a consequence of Lemma 6.14 (with a replaced by $\tilde a$) and Lemma 6.2.3. □

In conclusion, we have

Proposition 6.2.5.

$$R_{m',n'}[l'_1, l'_2, l'_3] = \bigsqcup_{\#(I)=a, \#(J)=b} R_{m',n'}[l_1]^{I,J}. \tag{6.31}$$

Proposition 6.2.6.

$$R^{(M,N-1)}_{m',n'}[l'_1, l'_2, l'_3] = \bigsqcup_{\#(I)=a, \#(J)=b} R^{(M,N-1)}_{m',n'}[l_1]^{I,J}. \tag{6.32}$$

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